Max Flow/Min Cut
Ford-Fulkerson

\[(f \uparrow f')(u,v) = \text{flow } f \text{ augmented by } f'\]

\[(f \uparrow f')(u,v) = f(u,v) + f'(u,v) - f'(v,u)\]

Lemma 26.1: Let \(f\) be the flow in \(G\), and \(f'\) be a flow in \(G_f\), then \((f \uparrow f')\) is a flow in \(G\) with total amount:

\[|f \uparrow f'| = |f| + |f'|\]

Proof: pages 718-719
Ford-Fulkerson

For some path $p$:
$c_f(p) = \min(c_f(u,v) : (u,v) \text{ on } p)$

\(^\wedge\wedge\) (capacity of path is smallest edge)

Claim 26.3:
Let $f_p = f_p(u,v) = c_f(p)$, then
$|f \uparrow f_p| = |f| + |f_p|$
Ford-Fulkerson

Ford-Fulkerson(G, s, t)
for: each edge (u,v) in G.E: (u,v).f=0
while: exists path from s to t in G_f
  find c_f(p) // minimum edge cap.
  for: each edge (u,v) in p
    if(u,v) in E: (u,v).f=(u,v).f + c_f(p)
    else: (u,v).f=(u,v).f - c_f(p)
Ford-Fulkerson

Runtime:

How hard is it to find a path?

How many possible paths could you find?
Ford-Fulkerson

Runtime:

How hard is it to find a path?
- $O(E)$ (via BFS or DFS)

How many possible paths could you find?
- $|f^*|$ (paths might use only 1 flow)

.... so, $O(E |f^*|)$
Max flow, min cut

Relationship between cuts and flows?

\[ c(S,T) = \sum_{u \in S} \sum_{v \in T} c(u,v) \]

\[ f(S,T) = \sum_{u \in S} \sum_{v \in T} f(u,v) - \sum u \sum v f(v,u) \]
Max flow, min cut

Relationship between cuts and flows?

\[ c(S, T) = \sum_{u \in S} \sum_{v \in T} c(u, v) \]
\[ f(S, T) = \sum_{u \in S} \sum_{v \in T} f(u, v) - \sum_u \sum_v f(v, u) \]

cuts $\geq$ flows
Lemma 26.4
Let \((S,T)\) be any cut, then \(f(S,T) = |f|\)

Proof:
Page 722
(Again, kinda long)
Corollary 26.5
Flow is not larger than cut capacity
Proof:
\[ |f| = \sum_{u \in S} \sum_{v \in T} f(u,v) - \sum_{u} \sum_{v} f(v,u) \]
\[ \leq \sum_{u \in S} \sum_{v \in T} f(u,v) \]
\[ \leq \sum_{u \in S} \sum_{v \in T} c(u,v) \]
\[ = c(S,T) \]
Theorem 26.5
All 3 are equivalent:
1. $f$ is a max flow
2. Residual network has no aug. path
3. $|f| = c(S,T)$ for some cut $(S,T)$

Proof:
Will show: $1 \Rightarrow 2$, $2 \Rightarrow 3$, $3 \Rightarrow 1$
Max flow, min cut

f is a max flow $\Rightarrow$ Residual network has no augmenting path

Proof:
Assume there is a path $p$

$|f \uparrow f_p| = |f| + |f_p| > |f|$, which is a contradiction to $|f|$ being a max flow
Max flow, min cut

Residual network has no aug. path $\Rightarrow$ $|f| = c(S,T)$ for some cut $(S,T)$

Proof:
Let $S =$ all vertices reachable from $s$ in $G_f$
$u \in S, v \in T \Rightarrow f(u,v) = c(u,v)$ else there would be path in $G_f$
Max flow, min cut

Also, \( f(v,u) = 0 \) else \( c_f(u,v) > 0 \) and again \( v \) would be reachable from \( s \)

\[
f(S,T) = \sum_{u \in S} \sum_{v \in T} f(u,v) - \sum_{u} \sum_{v} f(v,u)
\]

\[
= \sum_{u \in S} \sum_{v \in T} c(u,v) - \sum_{u} \sum_{v} 0
\]

\[
= c(S,T)
\]
Max flow, min cut

$|f| = c(S,T)$ for some cut $(S,T)$

$\Rightarrow f$ is a max flow

Proof:

$|f| \leq c(S,T)$ for all cuts $(S,T)$

Thus trivially true
Edmonds-Karp
exists shortest path (BFS)

Ford-Fulkerson(G, s, t)
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while: exists path from s to t in G_f
find c_f(p) // minimum edge cap.
for: each edge (u,v) in p
  if(u,v) in E: (u,v).f=(u,v).f + c_f(p)
  else: (u,v).f=(u,v).f - c_f(p)
Edmonds-Karp

Lemma 26.7
Shortest path in $G_f$ is non-decreasing

Theorem 26.8
Number of flow augmentations by Edmonds-Karp is $O(|V||E|)$

Running time: $O(|V||E|^2)$
Matching

Another application of network flow is maximizing matchings in a bipartite graph.

Each node cannot be “used” twice.
Matching

Add “super sink” and “super source” (and direct edges source -> sink) capacity = 1 on all edges