Weighted graphs

Your mother is so fat,
even I cannot find the shortest path around her.
Announcements

Programming assignment DONE

Midterm next Tues
- Covers sorting, selection
Weighted graph

Edges in weighted graph are assigned a weight: \( w(v_1, v_2), v_1, v_2 \) in \( V \)

If path \( p = <v_0, v_1, \ldots, v_k> \) then the weight is: 
\[
    w(p) = \sum_{i=0}^{k} (v_{i-1}, v_i)
\]

Shortest Path:
\[
    \delta(u, v): \min \{ w(p) : v_0 = u, v_k = v \}
\]
Shortest paths

Today we will look at single-source shortest paths.

This finds the shortest path from some starting vertex, $s$, to any other vertex on the graph (if it exists).

This creates $G_\pi$, the shortest path tree.
Shortest paths

Optimal substructure: Let $\delta(v_0,v_k)=p$, then for all $0 \leq i \leq j \leq k$, $\delta(v_i,v_j)=p_{i,j} = <v_i, v_{i+1}, \ldots, v_j>

Proof?

Where have we seen this before?
Shortest paths

Optimal substructure: Let $\delta(v_0,v_k)=p$, then for all $0 \leq i < j \leq k$, $\delta(v_i,v_j)=p_{i,j} = <v_i, v_{i+1}, \ldots, v_j>$

Proof? Contradiction! Suppose $w(p'_{i,j}) < p_{i,j}$, then let $p'_{0,k} = p'_{0,i} p'_{i,j} p_{j,k}$, then $w(p'_{0,k}) < w(p)$
Relaxation

We will only do relaxation on the values v.d (min weight) for vertex v

Relax(u,v,w)
if(v.d > u.d + w(u,v))
    v.d = u.d+w(u,v)
    v.π=u
Relaxation

We will assume all vertices start with \( v.d=\infty, v.\pi=\text{NIL} \) except \( s \), \( s.d=0 \)

This will take \( O(|V|) \) time

This will not effect the asymptotic runtime as it will be at least \( O(|V|) \) to find single-source shortest path
Relaxation

Relaxation properties:
1. $\delta(s,v) \leq \delta(s,u) + \delta(u,v)$ (triangle inequality)
2. $v.d \geq \delta(s,v)$, $v.d$ is monotonically decreasing
3. if no path, $v.d = \delta(s,v) = \infty$
4. if $\delta(s,v)$, when $(v.\pi).d = \delta(s,v.\pi)$ then relax$(v.\pi,v,w)$ causes $v.d = \delta(s,v)$
5. if $\delta(v_0,v_k) = p_{0,k}$, then when relaxed in order $(v_0, v_1), (v_1, v_2), \ldots (v_{k-1}, v_k)$ then $v_k.d = \delta(v_0, v_k)$ even if other relax happen
6. when $v.d = \delta(s,v)$ for all $v$ in $V$, $G_\pi$ is shortest path tree rooted at $s$
Directed Acyclic Graphs

DFS can do topographical sort (DAG)

Run DFS, sort in decreasing finish time
Directed Acyclic Graphs

DAG-shortest-paths(G,w,s)
topologically sort G
initialize graph from s
for each u in V in topological order
    for each v in G.Adj[u]
        Relax(u,v,w)

Runtime: $O(|V| + |E|)$
Depth first search
Directed Acyclic Graphs

Correctness:

Prove it!
Correctness:
By definition of topological order, When relaxing vertex v, we have already relaxed any preceding vertices

So by relaxation property 5, we have found the shortest path to all v
BFS (unweighted graphs)

Create FIFO queue to explore unvisited nodes
Dijkstra

Dijkstra's algorithm is the BFS equivalent for non-negative weight graphs
Dijkstra

Dijkstra(G,w,s)
initialize G from s
Q = G.V, S = empty
while Q not empty
    u = Extract-min(Q)
    S = S U {u}
    for each v int G.Adj[u]
        relax(u,v,w)

S optional
Dijkstra

Runtime?
Dijkstra

Runtime:
Extract-min() run $|V|$ times
Relax runs Decrease-key() $|E|$ times
Both take $O(lg n)$ time

So $O((|V| + |E|) lg |V|)$ time
(can get to $O(|V|lg|V| + E)$ using Fibonacci heaps)
Dijkstra

Runtime note:
If $G$ is almost fully connected, 
$|E| \approx |V|^2$

Use a simple array to store $v.d$
$\text{Extract-min}() = O(|V|)$
$\text{Decrease-key}() = O(1)$
total: $O(|V|^2 + E)$
Dijkstra

Correctness: (p.660)
Sufficient to prove when u added to S, u.d = δ(s,u)

Base: s added to S first, s.d=0=δ(s,s)

Termination: Loop ends after Q is empty, so V=S and we done
Step: Assume v in S has v.d = \( \delta(s,v) \)
Let y be the first vertex outside S on path of \( \delta(s,u) \)

We know by relaxation property 4, that \( \delta(s,y) = y.d \) (optimal sub-structure)

\[ y.d = \delta(s,y) \leq \delta(s,u) \leq u.d, \text{ as } w(p) \geq 0 \]
Dijkstra

Step: Assume \( v \) in \( S \) has \( v.d = \delta(s,v) \)
But as \( u \) was picked before \( y \),
\( u.d \leq y.d \), combined with \( y.d \leq u.d \)

\( y.d = u.d \)

Thus \( y.d = \delta(s,y) = \delta(s,u) = u.d \)