Assignment #7: Determinants, Eigenvalues and Eigenvectors

Due date: Wednesday, November 2, 2016 (9:10am)

Name: ____________________________________________________________

Section Number


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1. (8 points) Compute the determinants of the following matrices, using the correspondingly most efficient of the various different approaches described in class. Try to use a different approach to solve each one. You can use Matlab to check your answers. Please show all of your work.

\[
\begin{align*}
a. & \quad \begin{bmatrix} 1 & 5 & 9 & 7 \\ 0 & -2 & 0 & -1 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} & \quad b. & \quad \begin{bmatrix} 0 & 0 & 1 & 2 \\ 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 2 & 3 \end{bmatrix} & \quad c. & \quad \begin{bmatrix} 3 & 7 & 1 & 4 \\ 2 & 1 & 2 & 5 \\ 3 & 7 & 1 & 4 \\ 2 & 1 & 2 & 5 \end{bmatrix} & \quad d. & \quad \begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & 0 & 2 \\ 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 3 \end{bmatrix}
\end{align*}
\]

2. (12 points) Compute the determinants of the following matrices, using the correspondingly most efficient of the various different approaches described in class. Try to use a different approach to solve each one. Please justify your choice in each case.

\[
\begin{align*}
a. & \quad \begin{bmatrix} a & c & b & d \\ 0 & b & 0 & c \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & d \end{bmatrix} & \quad b. & \quad \begin{bmatrix} 0 & 0 & a & b \\ a & b & c & d \\ 0 & 0 & 0 & a \\ 0 & a & b & c \end{bmatrix} & \quad c. & \quad \begin{bmatrix} a & b & c & d \\ e & f & g & h \\ a & b & c & d \\ e & f & g & h \end{bmatrix} & \quad d. & \quad \begin{bmatrix} a & 0 & c & 0 \\ 0 & a & 0 & b \\ c & 0 & 0 & 0 \\ 0 & b & 0 & c \end{bmatrix}
\end{align*}
\]

3. (5 points) If \( A \) is an \( n \times n \) matrix, and \( \det A = d \), what is \( \det(kA) \)? Please justify your answer.

4. (6 points) If \( \det A = d \), where \( d \neq 0 \), what is \( \det(A^{-1}) \)? Please show all of your work. Hint: consider the rule that \( \det(AB) = (\det(A))(\det(B)) \).

5. (5 points) Suppose \( B = PAP^{-1} \). Show that \( \det B = \det A \).

6. (12 points) Write a function in Matlab to generate a random \( m \times n \) matrix \( A \), where \( m \) and \( n \) are input parameters and the elements of \( A \) are integers between \(-5\) and \( 5\). Then, write a program in Matlab that uses this function to compute \( \det(\mathbf{A}A^T) \) and \( \det(A^T\mathbf{A}) \) for 8 different random \( 3 \times 5 \) matrices. What do you notice? Explain your findings. Please pass in a printout of your program and its output.

7. (14 points) Let \( A = \begin{bmatrix} 4 & 0 & -7 & -7 \\ -6 & 1 & 11 & 9 \\ 7 & -5 & 10 & 19 \\ -1 & 2 & 3 & -1 \end{bmatrix} \).

a. Use Matlab to compute the rank of \( A \), the determinant of \( A \) and the condition number of \( A \).

b. Use Matlab to compute \( A^{-1} \), \( I - AA^{-1} \) and \( I - A^{-1}A \). What do you notice?

c. Compare the condition number of \( A \) to the condition number of \( H \) where \( H \) is a \( 4 \times 4 \) Hilbert matrix (a canonical example of an ill-conditioned matrix).
d. Let $B = 100^*A$, and let $C = 0.01^*A$. Compute $B^{-1}$, $I - BB^{-1}$, $I - B^{-1}B$, $C^{-1}$, $I - CC^{-1}$, $I - C^{-1}C$. What do you notice?

e. Compute the determinants and condition numbers of $B$ and $C$, and compare them to the determinant and condition number of $A$.

f. What do you conclude about the usefulness of the determinant as a metric for indicating how close a matrix is to being singular?

8. (16 points) For each of the following matrices: i) describe the geometric operation that each matrix represents; ii) describe the eigenvectors and the subspaces they span; iii) for each eigenspace, specify the associated eigenvalue. Please justify all of your answers using geometric reasoning.

a. $\text{Sh} = \begin{bmatrix} 1 & 0 \\ sh_y & 1 \end{bmatrix}$

b. $\text{Rf} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

c. $\text{P} = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$

d. $\text{R} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$

9. (8 points) Let $A = \begin{bmatrix} 8 & 7 & 5 \\ 2 & 3 & 2 \\ 1 & 6 & 4 \end{bmatrix}$, $u = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ and $v = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Is $u$ an eigenvector of $A$? Is $v$ an eigenvector of $A$? Please show all of your work.

10. (8 points) Let $A = \begin{bmatrix} 5 & -6 & 12 \\ 2 & -2 & 7 \\ 0 & 0 & -1 \end{bmatrix}$. Is 5 an eigenvalue of $A$? Is $-1$ an eigenvalue of $A$? Please show all of your work.

11. (6 points)

a. True or False. If $\lambda$ is a non-zero eigenvalue of an invertible matrix $A$ then $1/\lambda$ is an eigenvalue of $A^{-1}$.

b. True or False. For every square matrix $A$, $A$ and $A^T$ have the same eigenvalues.

c. True or False. For every square matrix $A$, $A$ and $A^T$ have the same eigenvectors.