Assignment #11: Orthogonal Projections, Gram-Schmidt, and Least Squares

Due date: Wednesday, December 7, 2016 (9:10am)

Name: __________________________________________

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For full credit you must show all of your work.

1. If \( \mathbf{v}_1 = \begin{bmatrix} 1 \\ 3 \\ 3 \\ 1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} -1 \\ 3 \\ -3 \\ 1 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} -3 \\ 1 \\ 1 \\ -3 \end{bmatrix} \) and \( \mathbf{v}_4 = \begin{bmatrix} 3 \\ 1 \\ -1 \\ -3 \end{bmatrix} \) form an orthogonal basis for \( \mathbb{R}^4 \),

   a. (4 points) Find the orthogonal projection of \( \mathbf{x} = \begin{bmatrix} 1 \\ 6 \\ 6 \\ 8 \\ 2 \end{bmatrix} \) onto each of the 1D subspaces of \( \mathbb{R}^4 \) spanned by each of the basis vectors \( \mathbf{v}_i \)

   b. (2 points) Express \( \mathbf{x} = \begin{bmatrix} 16 \\ 6 \\ 8 \\ 2 \end{bmatrix} \) as a linear combination of \{\( \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4 \}\).

   c. (2 points) Find the closest point \( \hat{\mathbf{x}} \) to \( \mathbf{x} \) in the subspace \( W \) of \( \mathbb{R}^4 \) spanned by \{\( \mathbf{v}_3, \mathbf{v}_4 \)\}.

   d. (2 points) Express \( \mathbf{x} \) as the sum of two orthogonal vectors, \( \mathbf{u} \) which is in the subspace \( W \) spanned by \{\( \mathbf{v}_1, \mathbf{v}_2 \)\} and \( \mathbf{w} \) which is in \( W^\perp \).

2. Let \( \mathbf{v}_1 = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} \) and \( \mathbf{v}_2 = \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix} \) be an orthogonal basis for a 2D subspace of \( \mathbb{R}^3 \) and let \( \mathbf{V} \) be the matrix whose columns are defined by \{\( \mathbf{v}_1, \mathbf{v}_2 \)\}.

   a. (3 points) Is \( \mathbf{V} \) an orthogonal matrix? Compute \( \mathbf{A} = \mathbf{V}^T \mathbf{V} \).

   b. (2 points) Let \( \mathbf{U} \) be a matrix with orthonormal columns \( \mathbf{u}_1 \) and \( \mathbf{u}_2 \) that span the same subspace of \( \mathbb{R}^3 \) as is spanned by the columns of \( \mathbf{V} \). Is \( \mathbf{U} \) an orthogonal matrix? Compute \( \mathbf{B} = \mathbf{U}^T \mathbf{U} \).

   c. (2 points) Compute \( \mathbf{w} = (\mathbf{U} \mathbf{U}^T) \mathbf{y} \) where \( \mathbf{y} = \begin{bmatrix} 9 \\ 0 \\ 0 \end{bmatrix} \).

   d. (3 points) Is it possible to express \( \mathbf{w} \) as a linear combination of \( \mathbf{u}_1 \) and \( \mathbf{u}_2 \)? Is it possible to express \( \mathbf{y} \) as a linear combination of \( \mathbf{u}_1 \) and \( \mathbf{u}_2 \)? How do you know?

   e. (2 points) Express \( \mathbf{y} \) as the sum of two orthogonal vectors, one of which is in the subspace spanned by \( \mathbf{u}_1 \) and \( \mathbf{u}_2 \).
f. (4 points) Find an orthogonal matrix $Q$ whose first two columns are $u_1$ and $u_2$. Compute $Q^TQ$ and $QQ^T$.

3. (6 points) Use the Gram-Schmidt Process to find an orthogonal basis for the subspace of $\mathbb{R}^4$ spanned by $u_1 = \begin{bmatrix} 1 \\ 1 \\ -3 \\ 1 \end{bmatrix}$, $u_2 = \begin{bmatrix} 5 \\ 1 \\ -7 \\ 9 \end{bmatrix}$, and $u_3 = \begin{bmatrix} -1 \\ -1 \\ 0 \\ 5 \end{bmatrix}$.

4. (3 points) What complications can arise when you try to use the Gram-Schmidt process to find an orthogonal basis for the column space of an $m \times n$ matrix whose columns are linearly dependent? Hint: try it on the following matrices: $A = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 & 2 \\ 0 & 0 & 1 \end{bmatrix}$.

5. (12 points) Using Matlab, write a function `GramSchmidt()` that takes as input an arbitrary $m \times n$ matrix $M$ and produces as output an $m \times p$ matrix $B$ whose columns form an orthogonal basis for the subspace of $\mathbb{R}^m$ spanned by the columns of $M$. You do not need to normalize the columns of $B$. Test your program on the input from questions 3 and 4 above and verify that your program produces valid results. Please pass in a printout of your code and its output.

6. (5 points) Working by hand, find the QR decomposition of $A = \begin{bmatrix} 1 & 2 \\ 2 & -2 \\ -2 & 8 \end{bmatrix}$, where $Q$ is a 3 x 2 matrix whose columns form an orthonormal basis for the column space of $A$ and $R$ is a 2 x 2 invertible, upper triangular matrix such that $A = QR$ (in other words, $R = Q^T A$).

7. (5 points) Working by hand, find the QR decomposition of $A = \begin{bmatrix} 1 & 2 \\ 2 & -2 \\ -2 & 8 \end{bmatrix}$, where $Q$ is a 3 x 3 orthogonal matrix and $R$ is a 3 x 2 upper triangular matrix such that $A = QR$ (in other words, $R = Q^T A$).

8. (6 points) What is the Householder reflection matrix $Q$ that will rotate the vector $x = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ into alignment with $e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$?

9. (8 points) Find the best-fitting line (in the least-squares sense) to these four points: $(-1, 1), (0, 3), (1, 4), (2, 4)$. Use Matlab or an equivalent graphing program to visualize your results, plotting both the original points and the fitted line. What is the least squares error in this approximation?
10. (12 points) Using Matlab, write a function `LinearFit()` that takes as input an arbitrary number of 2D points \((x, y)\), expressed as an \(m \times 2\) matrix where \(m\) is the total number of points, and uses the least-squares method to determine the equation of the “best-fitting” line to those points, expressed in terms of the slope and intercept. Use Matlab to plot the points and the line. Test your code using the data from question 9 above. Please submit an electronic version of your code via Moodle.

11. (8 points) Find the best-fitting quadratic (in the least-squares sense) to these four points: \((-2, 2), (-1, -2), (1, 0), (2, 2)\). Use Matlab or an equivalent graphing program to visualize your results, plotting both the original points and the fitted curve. For full credit you must show all of your work. You can check your answers using the Matlab function `polyfit()`.

12. (6 points) Find the best-fitting plane (in the least-squares sense) to these four points: \((1, 2, 1), (2, 0, 2), (0, 1, 3), (-1, -1, -1)\). For full credit you must show all of your work. You can check your answer using Matlab: `coeffs = [x y c] \ z` where \(x, y, z\) are vectors containing the \(x, y, z\) coordinates of the input data points and \(c\) is a vector of the same length containing all 1s.

13. (3 points) Which of the systems listed below has a unique least squares solution? Explain how you can tell without calculating the least squares solution.

\[
\begin{bmatrix}
1 & -1 & 1 \\
2 & -2 & 3 \\
2 & -2 & 3 \\
1 & -1 & 1
\end{bmatrix} \begin{bmatrix}
x \\
y \\
z
\end{bmatrix} = \begin{bmatrix}
2 \\
1 \\
1 \\
2
\end{bmatrix}
\]

\[
\begin{bmatrix}
1 & 2 & 2 & 1 \\
1 & 3 & 3 & 1 \\
2 & 3 & 1 & 3
\end{bmatrix} \begin{bmatrix}
x \\
y \\
z
\end{bmatrix} = \begin{bmatrix}
1 \\
2 \\
1
\end{bmatrix}
\]

\[
\begin{bmatrix}
1 & 1 & 2 \\
0 & -2 & 1 \\
0 & 1 & 3 \\
0 & 0 & -1
\end{bmatrix} \begin{bmatrix}
x \\
y \\
z
\end{bmatrix} = \begin{bmatrix}
0 \\
1 \\
2 \\
3
\end{bmatrix}
\]