Assignment #12: Diagonalization of Symmetric Matrices, Quadratic Forms, Optimization, Singular Value Decomposition

*Due date: Wednesday, December 14, 2016 (9:10am)*

Name: ____________________________________________________________

Section Number
Assignment #12: Diagonalization of Symmetric Matrices, Quadratic Forms, Optimization, Singular Value Decomposition

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For full credit you must show all of your work.

1. (10 points) Orthogonally diagonalize \( A = \begin{bmatrix} 1 & 4 & -8 \\ 4 & 7 & 4 \\ -8 & 4 & 1 \end{bmatrix} \).

2. (8 points) Suppose the symmetric matrix \( B = \begin{bmatrix} -1 & 8 & -4 \\ 8 & -1 & -4 \\ -4 & -4 & -7 \end{bmatrix} \) can be orthogonally diagonalized as

\[
B = \begin{bmatrix} -1/3 & -2/3 & 2/3 \\ 2/3 & 1/3 & 2/3 \\ 2/3 & -2/3 & -1/3 \end{bmatrix} \begin{bmatrix} -9 & 0 & 0 \\ 0 & -9 & 0 \\ 0 & 0 & 9 \end{bmatrix} \begin{bmatrix} -1/3 & 2/3 & 2/3 \\ -2/3 & 1/3 & -2/3 \\ 2/3 & 2/3 & -1/3 \end{bmatrix}
\]

What is the spectral decomposition of \( B \)?

3.

a. (8 points) Let \( v = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \) be any arbitrary vector in \( \mathbb{R}^3 \)

i. Compute the matrix \( M = vv^T \)

ii. Verify that \( M \) is symmetric

iii. Verify that rank(\( M \)) = 1

iv. Verify that \( v \) is an eigenvector of \( M \): in other words, that \( Mv = \lambda v \).

v. What is the associated eigenvalue \( \lambda \) (either in terms of \( v \) or its components \( a, b, c \))?

b. (6 points) Let \( w = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \) be any arbitrary vector in \( \mathbb{R}^3 \).

i. Verify that the matrix \( M = vv^T \) projects \( w \) onto the subspace of \( \mathbb{R}^3 \) spanned by \( v \).
   Specifically: \( M(x) = \alpha v \).

ii. Show that \( \alpha = (v \cdot w) \).

c. (4 points) Show that for any \( m \times n \) matrix \( A \), the matrix products \( A^TA \) and \( AA^T \) are symmetric.
4. (12 points) Construct the matrix of the quadratic form \( Q(x) = 11x_1^2 - 4x_1x_2 + 2x_2^2 + 16x_3x_5 + 20x_2x_3 + 5x_3^2 \). Use a change-of-variable to express the quadratic form of \( A \) without using any cross-product terms. Classify this quadratic form. What are its principal axes?

5. (5 points) For what values of \( a \) will the matrix \( A = \begin{bmatrix} a & 4 \\ 4 & 2 \end{bmatrix} \) be positive definite? positive semi-definite? indefinite? Are there any values of \( a \) for which this matrix will be negative definite?

6. (5 points) For what values of \( b \) will the matrix \( B = \begin{bmatrix} 4 & b \\ b & -2 \end{bmatrix} \) be negative definite? Are there any values of \( b \) for which this matrix will be indefinite? Please explain.

7. (8 points)
   a. Prove that for any arbitrary \( m \times n \) matrix \( A \), the matrix \( A^T A \) is guaranteed to be positive semi-definite.
   b. Under what conditions will the product \( A^T A \) of an arbitrary \( m \times n \) matrix \( A \) fail to be positive definite? (Characterize the matrices \( A \) for which this outcome will occur.) Please justify your answer.

8. (11 points) Consider the symmetric matrix \( B = \begin{bmatrix} -1 & 8 & -4 \\ 8 & -1 & -4 \\ -4 & -4 & -7 \end{bmatrix} \) which can be orthogonally diagonalized as:

\[
\begin{bmatrix}
-1/3 & -2/3 & 2/3 \\
2/3 & 1/3 & 2/3 \\
2/3 & -2/3 & -1/3
\end{bmatrix}
\begin{bmatrix}
-9 & 0 & 0 \\
0 & -9 & 0 \\
0 & 0 & 9
\end{bmatrix}
\begin{bmatrix}
-1/3 & 2/3 & 2/3 \\
-2/3 & 1/3 & -2/3 \\
2/3 & 2/3 & -1/3
\end{bmatrix}
\]

   a. What is the minimum of the quadratic form \( Q(x) \) of \( B \), over all unit length vectors \( x \)?
   b. What is the maximum of the quadratic form \( Q(x) \) of \( B \), over all unit length vectors \( x \)?
   c. For what unit vectors \( x \) will \( Q(x) \) take its maximum value?
   d. For what unit vectors \( x \) will \( Q(x) \) take its minimum value?
   e. Find a unit vector \( x \) at which \( Q(x) = 0 \). Describe the set of unit vectors \( x \) at which \( Q(x) = 0 \).

9. (15 points) Working by hand, compute the singular value decomposition of the following matrices. (This will be good practice for the exam).

   a. \( A = \begin{bmatrix} 2 & 3 \\ 4 & 6 \end{bmatrix} \)
   b. \( B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & -1 \end{bmatrix} \)
   c. \( C = \begin{bmatrix} 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix} \)
10. (8 points) Let \( A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & -2 & 1 \end{bmatrix} \). \( A \) is a linear transformation that maps vectors \( x \) in \( \mathbb{R}^3 \) into vectors \( b \) in \( \mathbb{R}^2 \). Consider the set of all possible vectors \( b = Ax \), where \( x \) is of unit length. What is the longest vector \( b \) in this set, and what unit length vector \( x \) is used to obtain it?

11. (10 points extra credit) The singular value decomposition of an \( m \times n \) matrix \( A = U\Sigma V^T \) can be expressed as a sum of \( r \) distinct matrices \( B_1 \ldots B_r \), where \( r = \min(m,n) \) and \( B_i = u_i \sigma_i v_i^T \), where \( u_i \) is the \( i \)th column of \( U \), \( \sigma_i \) is the \( i \)th diagonal element of \( \Sigma \) and \( v_i^T \) is the \( i \)th row of \( V \). If we treat an image as an \( m \times n \) matrix, and decompose it using a singular value decomposition, then the sum of all of the components would be equal to the original image. But what is the result if we take the sum of only the first few components? This Matlab exercise provides an opportunity for you to explore that question. Please follow the steps below:

a. Use the command
   \[
   A = \text{imread('testpat1.png')};
   \]
   to load the cameraman image into a matrix \( A \). \( A \) will have dimensions 512 x 512, and its entries will be 8-bit unsigned integers.

b. Use the command
   \[
   [U, S, V] = \text{svd(double(A))};
   \]
   to obtain the singular value decomposition of \( A \), where \( A = USV^T \). Note that you have to convert the values of \( A \) from int to double before you can apply Matlab’s \text{svd}() function to it.

c. In a loop of increasing values of \( k \), for example \( k = 1 : 2 : 100 \), reconstruct a “reduced” version of \( A \) using only the first \( k \) components of the singular value decomposition. Specifically, you need to form the product \( a = usv^T \) where \( u \) is a 512 x \( k \) matrix containing the first \( k \) columns of \( U \), \( s \) is a \( k \) x \( k \) matrix containing the first \( k \) diagonal values in \( S \), and \( v^T \) is a \( k \) x 512 matrix containing the first \( k \) rows of \( V^T \). You can use a command like: \( u = U(:, i) \) to extract the \( i \)th column from \( U \), and a command like: \( u = U(:, i:j) \) to extract a range of columns between the \( i \)th and the \( j \)th. Be careful to extract the correct portions of each component matrix. In your loop, use the commands \text{figure} and \text{imshow(uint8(a))} to display the series of reconstructed images. (You need to convert the values of the matrix \( a \) back to 8-bit unsigned integers before they can be displayed as an image.)

d. Report your impressions. How many components do you need to use before the image becomes recognizable? After how many components do you stop seeing much difference in the result?

e. Extract and plot the singular values in matrix \( S \). You can do this using the command:
   \[
   \text{plot(1:length(diag(S)),diag(S))};
   \]
   Do you notice a relationship between the sizes of the singular values and the pace of improvement in the quality of the reduced image?

Please pass in: a printout of your code; your answers to the questions above, and one of the reduced images that you obtained, along with an annotation of the number of components you used to construct it (the value \( k \)).