1. Let $A$ and $B$ two nonsingular matrices of size $n \times n$. Show that

$$fl(AB) = (A + E_A)B \quad \text{where} \quad |E_A| \leq \gamma_n |A| |B| |B^{-1}|$$

($\gamma_n$ defined in the notes) and derive a corresponding bound in which $B$ is perturbed. [This result shows the limitation of backward error analysis. In this case it is clear that Forward error analysis yields a ‘cleaner’ result].

2. (a) Determine the standard LU factorization of the matrix

on the right.
(b) Compute the determinant of $A$
(c) Compute the inverse of $A$.
(d) Repeat the above questions when partial pivoting is used, i.e., find the permutation matrix $P$ and the matrices $L, U$ such that $PA = LU$, compute the determinant of $A$ based on this factorization, and compute the inverse of $A$, based on this factorization.
(e) Use the answer from (d) to solve the system $Ax = b$ when $b = [-2, 2, 2]^T$

3. (a) Given a nonsingular matrix $A$, and two column vectors $u$ and $v$, prove that if $1 + v^T A^{-1} u \neq 0$ then $A + uv^T$ is invertible and its inverse is given by the so-called Sherman-Morrison formula:

$$(A + uv^T)^{-1} = A^{-1} - \frac{1}{1 + v^T A^{-1} u} A^{-1} uv^T A^{-1}$$

[Hint: First answer the following question. Let $u \in \mathbb{R}^n, v \in \mathbb{R}^n$ with $v^T u + 1 \neq 0$. Show that $I + uv^T$ is nonsingular. What is the inverse $I + uv^T$? Then If $B = A + uv^T$ where $A$ is nonsingular, write $B$ as $B = A(I + A^{-1} uv^T)$ ...]

(b) Let $A$ be a square nonsingular matrix and let $B$ the matrix obtained from $A$ by adding a perturbation $\alpha$ to $a_{11}$. Assuming $B$ is nonsingular, give an upper bound for $\|B^{-1}\|$ in two different ways: (a) by using the standard result on $\|(A + E)^{-1}\|$; (b) by using the expression of the inverse using the Sherman-Morrison formula.

(c) Suppose you have already solved a linear system with the matrix $A$ (and saved the L, U factors) and then you need to solve a linear system with the matrix $B = A + uv^T$. How would you proceed?
4. Using matlab, plot in a logarithmic scale the condition numbers $\kappa_2(H_n)$ for $n = 3 : 12$, where $H_n$ is the Hilbert matrix of dimension $n$. [Note: the matlab command `hilb(n)` generates the $n$-th Hilbert matrix.] Based on the plot give an approximate expression for the condition number $\kappa_2(H_n)$ as a function of $n$.

5. Consider the matrix $A$ shown on the right, where $I$ represents the $n \times n$ identity.
   (a) What is the inverse of $A$?
   (b) Show that $\kappa_F(A) = 2n + \|Z\|^2_F$.
   (c) Express the 1-norm condition number of $A$ in terms of the 1-norm of $Z$.
   (d) Express the $\infty$-norm condition number of $A$ in terms of the $\infty$-norm of $Z$.

6. Obtain a lower bound for $\kappa_1(A)$ (without computing $A^{-1}$) for the matrix $A$

\[
A = \begin{pmatrix}
1 & 2 & 3.001 \\
4 & 5.002 & 6 \\
6.999 & 8.001 & 8.999
\end{pmatrix}
\]

[Hint: $A$ is close to a singular matrix]

7. Show that $\kappa(AB) \leq \kappa(A)\kappa(B)$. Is it true in general that $\kappa(A) = \kappa(A^T)$? Show that $\kappa_2(A) = \kappa_2(A^T)$ and $\kappa_2(A^T A) = \kappa_2(A)^2$.

8. Consider the following two systems.

\[
Ax \equiv \begin{pmatrix}
1 & 1 & -1 \\
1 & 2 & 0.01 \\
0 & 1 & 1
\end{pmatrix} \begin{pmatrix}
x_1 \\
x_2 \\
x_3
\end{pmatrix} = \begin{pmatrix}
-1 \\
1.01 \\
2.0
\end{pmatrix}
\]

\[
\tilde{A}y \equiv \begin{pmatrix}
1 & 1 & -1 \\
1.0001 & 2 & 0.01 \\
0.0 & 1 & 1.0001
\end{pmatrix} \begin{pmatrix}
y_1 \\
y_2 \\
y_3
\end{pmatrix} = \begin{pmatrix}
-1 \\
1.01 \\
2.0
\end{pmatrix}
\]

The solution of the first system is $x_1 = -1; x_2 = x_3 = 1$. (a) Using matlab compute the solution $y$ to the second system and compute also $\kappa_\infty(A)$. (b) Then compute $\|x - y\|_\infty/\|x\|_\infty$ and an upper bound for it obtained using the condition number. (c) Finally, get an upper bound for the same quantity obtained using the componentwise condition number (theorem 4 from Lect. notes set 6). ← Leave this last question out for next homework