1. Consider the problem \( \min \| b - Ax \|_2 \) in the situation where \( A \) is \( m \times n \) and \( m < n \) (‘underdetermined’ case). Assume that \( \text{rank}(A) = m \). Using what you learned from the URV decomposition, show that the set of solutions is of dimension \( n - m \). Show that the least-squares solution \( x_\ast \) of smallest norm must belong to \( \text{Ran}(A^T) \). Find a method for computing \( x_\ast \) which involves a form of normal equations. In HW4, we saw a solution that involved the QR factorization. Now, find a method method for computing \( x_\ast \) based on the SVD.

2. For this exercise, you can do all calculations by hand, and use matlab to verify or to help. Consider the matrix:

\[
A = \begin{pmatrix}
-1 & 0 & -1 \\
0 & 1 & 1 \\
1 & 0 & 1 \\
0 & 0 & 0
\end{pmatrix}
\]

(a) What is the rank of \( A \)? Find orthonormal bases for the range of \( A \) and the range of \( A^T \).

(b) Complete these bases orthonormally into bases of the appropriate spaces and find an URV factorization of \( A \).

(c) What are the singular values of \( A \)? [You need to use your URV decomposition. Use matlab only to verify your result.]

3. Prove that if \( A \in \mathbb{R}^{m \times n} \) then

\[
\sigma_1(A) = \max_{y \in \mathbb{R}^m} \frac{y^T Ax}{\|x\|_2 \|y\|_2}
\]

[Hint: start by showing that \( \sigma_1 \geq \) the right-hand side by using the Cauchy-Schwartz inequality]

4. The nuclear norm is commonly used in machine learning. For a given matrix \( X \in \mathbb{R}^{m \times n} \), with nonzero singular values \( \sigma_1, \ldots, \sigma_r \), it is defined as

\[
\|X\|_* = \sum_{i=1}^r \sigma_i.
\]

This is also known as the Ky-Fan norm. We will admit here that this is indeed a valid matrix norm. [The nuclear norm is the one norm of the singular values viewed as a vector. The
2-norm is the infinity norm of the same vector, and the Frobenius norm its euclidean norm. All these to the class of Schatten norms which are Hölder norms of the vector of singular values.

The purpose of this question is to show that

\[ \|A\|_* = \min_{U, V, s.t. UV^T = A} \frac{1}{2} \left( \|U\|_F^2 + \|V\|_F^2 \right) \]  

(1)

(a) Using the SVD of \( A \) show that the left-hand side of (1) is larger than or equal to the right-hand side.

(b) Given any two matrices \( X \) and \( Y \) both in \( \mathbb{R}^{m \times n} \) show that

\[ \text{Trace}(XY^T) \leq \frac{1}{2} \left( \|X\|_F^2 + \|Y\|_F^2 \right) \]

(c) **Bonus points** Finally, use (b) to show that for any \( U, V \) we have

\[ \|UV^T\|_* \leq \frac{1}{2} \left( \|U\|_F^2 + \|V\|_F^2 \right) \]

and complete the proof of (1).

5. We are given a square matrix \( A \) that has the SVD \( A = U \Sigma V^T \). Let \( U = [u_1, u_2, \ldots, u_n] \), \( V = [v_1, \ldots, v_n] \) and \( \Sigma = \text{diag}(\sigma_1, \ldots, \sigma_n) \). Prove that the \( 2n \) eigenvalues of the matrix:

\[ H = \begin{pmatrix} 0 & A^T \\ A & 0 \end{pmatrix} \]

are \( \pm \sigma_i \) with the corresponding unit eigenvectors \( \frac{1}{\sqrt{2}} \begin{bmatrix} v_i \\ \pm u_i \end{bmatrix} \). Extend to a general rectangular matrix \( A \).

6. This question involves a small matlab experiment. You will find in the matlab page of the class, the image of a “clown face” which is badly damaged by noise. [essentially adding large values to some random locations of the pixels.] Load the image with the command `load('clwn1')`. This will yield a matrix \( X1 \) of pixels which can be viewed (see picture next) with the commands: `imagesc(X1); colormap('gray').`

To smooth the image you will use matlab’s `svds` to compute the \( k \)-rank SVD approximation of the image and show the corresponding picture. Try a few values of \( k \) between 10 and 50 and
show 4 different results going from what you think is poor to better rendering of the image. Be aware that you will not get a perfect image but that you need to reach a compromise between noise and sharpness.

7. Consider the matrix shown on the right: (a) What are the nonzero singular values of $A$? 

(b) If $A = U\Sigma V^T$ is the SVD decomposition of $A$, what is the matrix $V$? What is $\Sigma$?

(c) Find the first two columns of the matrix $U$.

(d) Find the matrix of rank 1, that is the closest to $A$ in the 2-norm sense, i.e., the matrix $A_1$ which minimizes $\|A - B\|_2$ over all $4 \times 2$ matrices $B$ that are of rank 1.

8. Show by a suitable counter example that $(AB)^\dagger \neq B^\dagger A^\dagger$. [Hint: find $A \in \mathbb{R}^{1\times 2}$, $B \in \mathbb{R}^{2\times 1}$]