ERROR AND SENSITIVITY ANALYSIS FOR SYSTEMS OF LINEAR EQUATIONS

- Conditioning of linear systems.
- Estimating errors for solutions of linear systems
- Backward error analysis
- Relative element-wise error analysis
Perturbation analysis for linear systems \((Ax = b)\)

Question addressed by perturbation analysis: determine the variation of the solution \(x\) when the data, namely \(A\) and \(b\), undergoes small variations. Problem is **ill-conditioned** if small variations in data cause very large variation in the solution.
Let $E$, be an $n \times n$ matrix and $e_b$ be an $n$-vector.

“Perturb” $A$ into $A(\epsilon) = A + \epsilon E$ and $b$ into $b + \epsilon e_b$.

Note: $A + \epsilon E$ is nonsingular for $\epsilon$ small enough.

Why?

The solution $x(\epsilon)$ of the perturbed system is s.t.

$$(A + \epsilon E)x(\epsilon) = b + \epsilon e_b.$$ 

Let $\delta(\epsilon) = x(\epsilon) - x$. Then,

$$(A + \epsilon E)\delta(\epsilon) = (b + \epsilon e_b) - (A + \epsilon E)x = \epsilon (e_b - Ex)$$

$$\delta(\epsilon) = \epsilon (A + \epsilon E)^{-1}(e_b - Ex).$$
\( x(\epsilon) \) is differentiable at \( \epsilon = 0 \) and its derivative is

\[
x'(0) = \lim_{\epsilon \to 0} \frac{\delta(\epsilon)}{\epsilon} = A^{-1} (e_b - Ex).
\]

A small variation \([\epsilon E, \epsilon e_b]\) will cause the solution to vary by roughly \( \epsilon x'(0) = \epsilon A^{-1} (e_b - Ex) \).

The relative variation is such that

\[
\frac{\|x(\epsilon) - x\|}{\|x\|} \leq \epsilon \|A^{-1}\| \left( \frac{\|e_b\|}{\|x\|} + \|E\| \right) + O(\epsilon^2).
\]

Since \( \|b\| \leq \|A\| \|x\| \):

\[
\frac{\|x(\epsilon) - x\|}{\|x\|} \leq \epsilon \|A\| \|A^{-1}\| \left( \frac{\|e_b\|}{\|b\|} + \|E\| \right) + O(\epsilon^2)
\]
The quantity \( \kappa(A) = \| A \| \| A^{-1} \| \) is called the condition number of the linear system with respect to the norm \( \| . \| \). When using the \( p \)-norms we write:

\[
\kappa_p(A) = \| A \|_p \| A^{-1} \|_p
\]

- **Note:** \( \kappa_2(A) = \sigma_{max}(A)/\sigma_{min}(A) \) = ratio of largest to smallest singular values of \( A \). Allows to define \( \kappa_2(A) \) when \( A \) is not square.

- **Determinant** *is not* a good indication of sensitivity

- **Small eigenvalues** *do not* always give a good indication of poor conditioning.
**Example:** Consider, for a large $\alpha$, the $n \times n$ matrix

$$A = I + \alpha e_1 e_n^T$$

- Inverse of $A$ is:

$$A^{-1} = I - \alpha e_1 e_n^T$$

- For the $\infty$-norm we have

$$\|A\|_\infty = \|A^{-1}\|_\infty = 1 + |\alpha|$$

so that

$$\kappa_\infty(A) = (1 + |\alpha|)^2.$$  

- Can give a very large condition number for a large $\alpha$ – but all the eigenvalues of $A$ are equal to one.
Rigorous norm-based error bounds

- Previous bound is valid only when perturbation is “small enough,” where “small” is not precisely defined.
- New bound valid within an explicitly given neighborhood.

**THEOREM 1:** Assume that $(A + E)y = b + e_b$ and $Ax = b$ and that $\|A^{-1}\|\|E\| < 1$. Then $A + E$ is nonsingular and

$$\frac{\|x - y\|}{\|x\|} \leq \frac{\|A^{-1}\| \|A\|}{1 - \|A^{-1}\| \|E\|} \left(\frac{\|E\|}{\|A\|} + \frac{\|e_b\|}{\|b\|}\right)$$

- To prove, first need to show that $A + E$ is nonsingular if $A$ is nonsingular and $E$ is small.
Begin with simple case:

**Lemma:** If \( \| E \| < 1 \) then \( I - E \) is nonsingular and
\[
\| (I - E)^{-1} \| \leq \frac{1}{1 - \| E \|}
\]

**Proof** is based on following 5 steps

a) Show: If \( \| E \| < 1 \) then \( I - E \) is nonsingular

b) Show: \((I - E)(I + E + E^2 + \cdots + E^k) = I - E^{k+1}\).

c) From which we get:
\[
(I - E)^{-1} = \sum_{i=0}^{k} E^i + (I - E)^{-1} E^{k+1} \rightarrow
\]
d) \((I - E)^{-1} = \lim_{k \to \infty} \sum_{i=0}^{k} E^i\). We write this as
\[(I - E)^{-1} = \sum_{i=0}^{\infty} E^i\]
e) Finally:
\[
\| (I - E)^{-1} \| = \left\| \lim_{k \to \infty} \sum_{i=0}^{k} E^i \right\| = \lim_{k \to \infty} \left\| \sum_{i=0}^{k} E^i \right\|
\leq \lim_{k \to \infty} \sum_{i=0}^{k} \| E^i \| \leq \lim_{k \to \infty} \sum_{i=0}^{k} \| E \|^i
\leq \frac{1}{1 - \| E \|}
\]
Can generalize result:

**LEMMA:** If $A$ is nonsingular and $\|A^{-1}\| \|E\| < 1$ then $A + E$ is non-singular and

$$\|(A + E)^{-1}\| \leq \frac{\|A^{-1}\|}{1 - \|A^{-1}\| \|E\|}$$

Proof is based on relation $A + E = A(I + A^{-1}E)$ and use of previous lemma.

Now we can prove the theorem:

**THEOREM 1:** Assume that $(A + E)y = b + e_b$ and $Ax = b$ and that $\|A^{-1}\|\|E\| < 1$. Then $A + E$ is nonsingular and

$$\frac{\|x - y\|}{\|x\|} \leq \frac{\|A^{-1}\| \|A\|}{1 - \|A^{-1}\| \|E\|} \left( \frac{\|E\|}{\|A\|} + \frac{\|e_b\|}{\|b\|} \right)$$
Proof: From \((A + E)y = b + e_b\) and \(Ax = b\) we get
\((A + E)(y - x) = e_b - Ex\). Hence:

\[ y - x = (A + E)^{-1}(e_b - Ex) \]

Taking norms → \(\|y - x\| \leq \|(A + E)^{-1}\| [\|e_b\| + \|E\| \|x\|] \)

Dividing by \(\|x\|\) and using result of lemma

\[ \frac{\|y - x\|}{\|x\|} \leq \|(A + E)^{-1}\| [\|e_b\|/\|x\| + \|E\|] \]

\[ \leq \frac{\|A^{-1}\|}{1 - \|A^{-1}\|\|E\|} [\|e_b\|/\|x\| + \|E\|] \]

\[ \leq \frac{\|A^{-1}\|\|A\|}{1 - \|A^{-1}\|\|E\|} \left[ \|e_b\|/\|A\|\|x\| + \|E\|/\|A\| \right] \]

Result follows by using inequality \(\|A\|\|x\| \geq \|b\|\)…. QED
Simplification when $e_b = 0$:

\[
\frac{\|x - y\|}{\|x\|} \leq \frac{\|A^{-1}\| \|E\|}{1 - \|A^{-1}\| \|E\|}
\]

Simplification when $E = 0$:

\[
\frac{\|x - y\|}{\|x\|} \leq \|A^{-1}\| \|A\| \frac{\|e_b\|}{\|b\|}
\]

Slightly less general form: Assume that $\|E\|/\|A\| \leq \delta$ and $\|e_b\|/\|b\| \leq \delta$ and $\delta \kappa(A) < 1$ then

\[
\frac{\|x - y\|}{\|x\|} \leq \frac{2\delta \kappa(A)}{1 - \delta \kappa(A)}
\]

Show the above result
Another common form:

**THEOREM 2:** Let \((A + \Delta A)y = b + \Delta b\) and \(Ax = b\) where \(\|\Delta A\| \leq \epsilon \|E\|\), \(\|\Delta b\| \leq \epsilon \|e_b\|\), and assume that \(\epsilon \|A^{-1}\| \|E\| < 1\). Then

\[
\frac{\|x - y\|}{\|x\|} \leq \frac{\epsilon \|A^{-1}\| \|A\|}{1 - \epsilon \|A^{-1}\| \|E\|} \left(\frac{\|e_b\|}{\|b\|} + \frac{\|E\|}{\|A\|}\right)
\]

Result to be seen later are of this type.
Normwise backward error

We solve $Ax = b$ and find an approximate solution $y$

Question: Find smallest perturbation to apply to $A, b$ so that *exact* solution of perturbed system is $y$
Normwise backward error in just $A$ or $b$

Suppose we model entire perturbation in RHS $b$.

- Let $r = b - Ay$ be the residual. Then $y$ satisfies $Ay = b + \Delta b$ with $\Delta b = -r$ exactly.
- The relative perturbation to the RHS is $\frac{\|r\|}{\|b\|}$.

Suppose we model entire perturbation in matrix $A$.

- Then $y$ satisfies $\left( A + \frac{ry^T}{y^Ty} \right) y = b$
- The relative perturbation to the matrix is

$$\frac{\left\| \frac{ry^T}{y^Ty} \right\|_2}{\|A\|_2} = \frac{\|r\|_2}{\|A\| \|y\|_2}$$
For a given \( y \) and given perturbation directions \( E, e_b \), we define the Normwise backward error:

\[
\eta_{E,e_b}(y) = \min\{\epsilon \mid (A + \Delta A)y = b + \Delta b; \\
\text{for all } \Delta A, \Delta b \text{ satisfying: } \|\Delta A\| \leq \epsilon \|E\|; \\
\text{and } \|\Delta b\| \leq \epsilon \|e_b\| \}
\]

In other words \( \eta_{E,e_b}(y) \) is the smallest \( \epsilon \) for which

\[
(1) \left\{ (A + \Delta A)y = b + \Delta b; \\
\|\Delta A\| \leq \epsilon \|E\|; \quad \|\Delta b\| \leq \epsilon \|e_b\| \right. 
\]
\( y \) is given (a computed solution). \( E \) and \( e_b \) to be selected (most likely 'directions of perturbation for \( A \) and \( b \')).

Typical choice: \( E = A, \ e_b = b \)

Explain why this is not unreasonable

Let \( r = b - Ay \). Then we have:

**THEOREM 3:** \( \eta_{E,e_b}(y) = \frac{\|r\|}{\|E\|\|y\| + \|e_b\|} \)

Normwise backward error is for case \( E = A, e_b = b \):

\[ \eta_{A,b}(y) = \frac{\|r\|}{\|A\|\|y\| + \|b\|} \]
Show how this can be used in practice as a means to stop some iterative method which computes a sequence of approximate solutions to $Ax = b$.

Consider the $6 \times 6$ Vandermonde system $Ax = b$ where $a_{ij} = j^2(i-1)$, $b = A \ast [1, 1, \cdots, 1]^T$. We perturb $A$ by $E$, with $|E| \leq 10^{-10}|A|$ and $b$ similarly and solve the system. Evaluate the backward error for this case. Evaluate the forward bound provided by Theorem 2. Comment on the results.
Proof of Theorem 3

Let \( D \equiv \|E\|\|y\| + \|e_b\| \) and \( \eta \equiv \eta_{E,e_b}(y) \). The theorem states that \( \eta = \|r\|/D \). Proof in 2 steps.

First: Any \( \Delta A, \Delta b \) pair satisfying (1) is such that \( \epsilon \geq \|r\|/D \). Indeed from (1) we have (recall that \( r = b - Ay \))

\[
Ay + \Delta Ay = b + \Delta b \rightarrow r = \Delta Ay - \Delta b \rightarrow
\]

\[
\|r\| \leq \|\Delta A\|\|y\| + \|\Delta b\| \leq \epsilon(\|E\|\|y\| + \|e_b\|) \rightarrow \epsilon \geq \frac{\|r\|}{D}
\]

Second: We need to show an instance where the minimum value of \( \|r\|/D \) is reached. Take the pair \( \Delta A, \Delta b \):

\[
\Delta A = \alpha rz^T; \quad \Delta b = \beta r \quad \text{with} \quad \alpha = \frac{\|E\|\|y\|}{D}; \quad \beta = \frac{\|e_b\|}{D}
\]
The vector $z$ depends on the norm used - for the 2-norm: $z = y/\|y\|^2$. Here: Proof only for 2-norm

a) We need to verify that first part of (1) is satisfied:

$$(A + \Delta A)y = Ay + \alpha r \frac{y^T}{\|y\|^2} y = b - r + \alpha r$$

$$= b - (1 - \alpha)r = b - \left(1 - \frac{\|E\|\|y\|}{\|E\|\|y\| + \|e_b\|}\right)r$$

$$= b - \frac{\|e_b\|}{D}r = b + \beta r \quad \rightarrow$$

$$(A + \Delta A)y = b + \Delta b \quad \leftarrow \text{The desired result}$$
b) Finally: Must now verify that $\|\Delta A\| = \eta \|E\|$ and $\|\Delta b\| = \eta \|e_b\|$. Exercise: Show that $\|uv^T\|_2 = \|u\|_2 \|v\|_2$

$$\|\Delta A\| = \frac{|\alpha|}{\|y\|^2} \|ry^T\| = \frac{\|E\| \|y\| \|r\| \|y\|}{D} \|y\|^2 = \eta \|E\|$$

$$\|\Delta b\| = |\beta| \|r\| = \frac{\|e_b\|}{D} \|r\| = \eta \|e_b\| \quad QED$$