CSCI 8980: Special Topics in Computer Science

Physics-Based Animation

18 — Fluid simulation on grids II

November 3, 2016
Particles vs. grids

**Particles**
- Mass, velocity, pressure, etc. stored on particles
- Particles move with fluid (*Lagrangian* representation)

**Grids**
- Velocity, pressure, etc. stored on grid vertices / cells
- Grid doesn’t move (*Eulerian* representation)
The fluid equations

\[ \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = \frac{\mathbf{f}_{\text{ext}}}{\rho} - \frac{1}{\rho} \nabla p + \nu \nabla^2 \mathbf{u} \]
The fluid equations

\[
\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = \frac{\mathbf{f}_{\text{ext}}}{\rho} + \frac{1}{\rho} \nabla p + \nu \nabla^2 \mathbf{u}
\]
Operator splitting

\[ u^n \]

Integrate \( \frac{\partial u}{\partial t} + u \cdot \nabla u = 0 \) for time \( \Delta t \)

Integrate \( \frac{\partial u}{\partial t} = \frac{1}{\rho} f^{\text{ext}} \) for time \( \Delta t \)

Integrate \( \frac{\partial u}{\partial t} = \frac{\mu}{\rho} \nabla^2 u \) for time \( \Delta t \)

Integrate \( \frac{\partial u}{\partial t} = -\frac{1}{\rho} \nabla p \) for time \( \Delta t \)

\[ u^{n+1} \]
Semi-Lagrangian advection
Semi-Lagrangian advection
Pressure

\[ \frac{\partial \mathbf{u}}{\partial t} = -\rho^{-1} \nabla p \]

For incompressible fluids, pressure is a constraint force

\[ \mathbf{J} \mathbf{v} = 0 \quad \Rightarrow \quad \mathbf{f}_c = \mathbf{J}^T \lambda \]

\[ \nabla \cdot \mathbf{u} = 0 \quad \Rightarrow \quad \mathbf{f}_p = -\nabla p \]

Compute \( p \) to enforce the constraint that \( \nabla \cdot \mathbf{u}^{n+1} = 0 \)

\[ (\rho^{-1} \Delta t) \nabla^2 p = \nabla \cdot \mathbf{u} \]
Staggered grids

Store pressure at cell centers, but velocities at cell faces

\[(\nabla^2 p)_{i,j} \approx \frac{(p_{i-1,j} + p_{i+1,j} + p_{i,j-1} + p_{i,j+1} - 4p_{i,j})}{\Delta x^2}\]

\[(\nabla \cdot \mathbf{u})_{i,j} \approx \frac{u_{i+1,j} - u_{i,j}}{\Delta x} + \frac{v_{i,j+1} - v_{i,j}}{\Delta x}\]

Finite differences line up:

- \(p, \nabla^2 p, \nabla \cdot \mathbf{u}\) at cell centers
- Components of \(\mathbf{u}\) and \(\nabla p\) at cell faces
Setting it up

Boundary conditions

• Solid obstacles: $\mathbf{u} \cdot \mathbf{n} = 0 \Rightarrow \nabla p \cdot \mathbf{n} = 0$

• Free surfaces: $p = 0$

$$\nabla^2 p = \left( \frac{\rho}{\Delta t} \right) \nabla \cdot \mathbf{u}$$

Poisson’s equation: find a scalar field whose Laplacian ($\nabla^2$) is prescribed.

• Build a linear system $\mathbf{A}p = \mathbf{b}$:
  rows of $\mathbf{A}$ contain Laplacian stencil / boundary conditions

• Solve for $p$, update $\mathbf{u}$
Pressure

\[ \nabla^2 p = \nabla \cdot \mathbf{u} \]

\[ \nabla \cdot \mathbf{u} = \frac{u_1 - u_0}{\Delta x} + \frac{v_1 - v_0}{\Delta x} \]

\[ \propto \text{ total amount of fluid flowing out of cell} \]

\[ \nabla^2 p = \frac{1}{\Delta x^2} (p_1 + p_2 + p_3 + p_4 - 4p) \]
Pressure

\[ \nabla^2 p = \frac{1}{\Delta x^2} (p_1 + p_2 + p_3 + p_4 - 4p) \]

\[ \mathbf{A} \mathbf{p} = [\nabla^2 p] \]

\[
\frac{1}{\Delta x^2} \begin{bmatrix}
1 & 1 & -4 & 1 & 1 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots \\
1 & \ddots & \ddots & \ddots & 1 \\
1 & \ddots & \ddots & \ddots & 1 \\
\end{bmatrix} \begin{bmatrix}
p_i \\
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\end{bmatrix} \]
<table>
<thead>
<tr>
<th></th>
<th>Triangle or tetrahedral meshes</th>
<th>Regular grids</th>
</tr>
</thead>
<tbody>
<tr>
<td>Lagrangian elastic bodies</td>
<td>✔</td>
<td>?</td>
</tr>
<tr>
<td>Eulerian fluids</td>
<td>?</td>
<td>✔</td>
</tr>
</tbody>
</table>
Next class

Student presentations


• Jung Nam: Barbič and James, “Real-Time Subspace Integration for St.Venant-Kirchhoff Deformable Models” (2005)

Everyone else: Read both papers and, for each one, submit a brief list of questions, comments, or issues for class discussion by Sunday 2 pm.