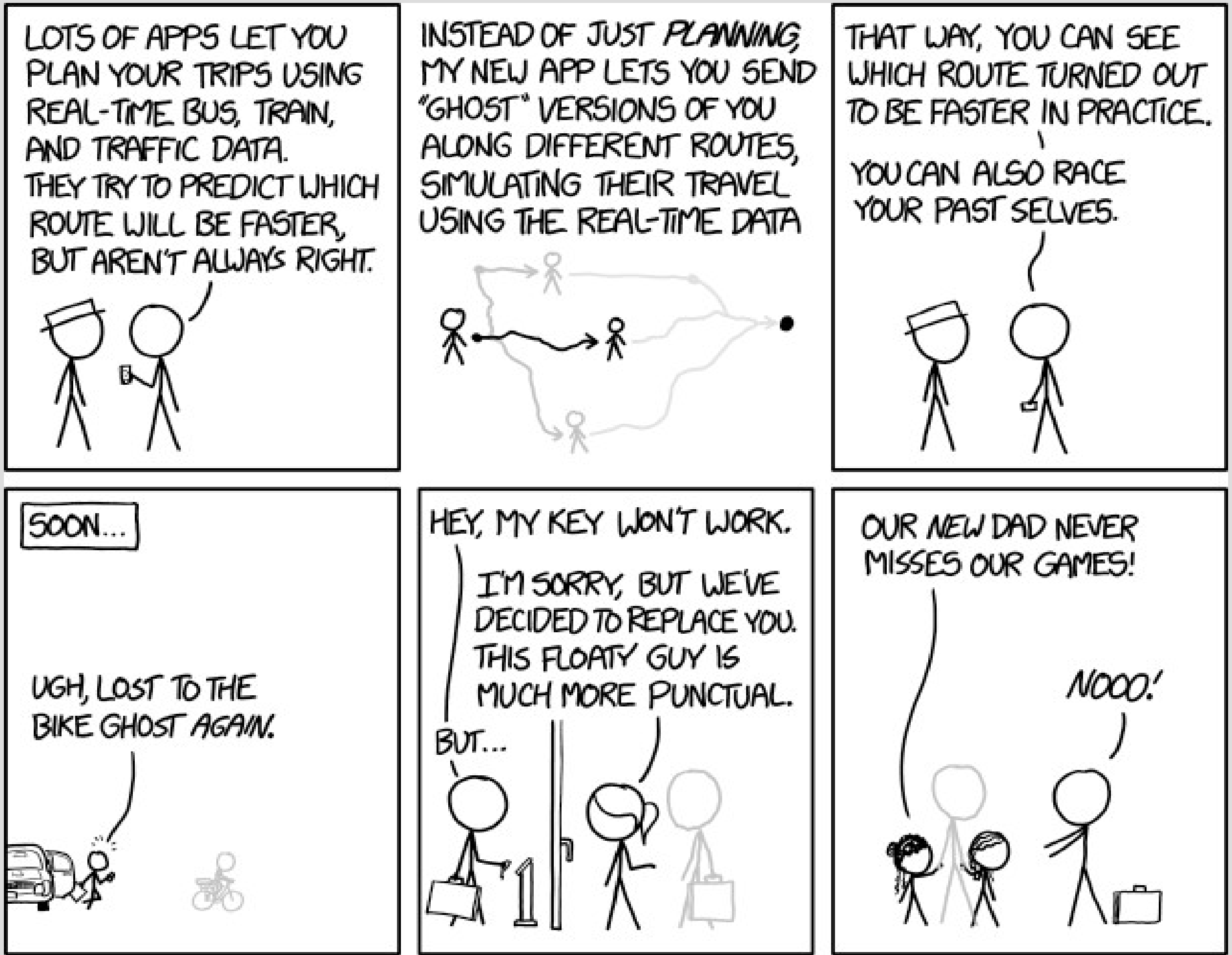
All-Pairs Shortest Paths



TL;DR dynamic programming

What are two ways you can compute the Fibonacci numbers?

$F_n = F_{n-1} + F_{n-2}$

with $F_0=0$, $F_1=1$

Which way is better?

TL;DR dynamic programming

One way, simply use the definition

Recursive:

```
F(n):
   if(n==1 or n==0)
      return n
   else return F(n-1)+F(n-2)
```

TL;DR dynamic programming

Another way, compute F(2), then F(3)
... until you get to F(n)

Bottom up:

```
A[0] = 0
A[1] = 1
for i = 2 to n
    A[i] = A[i-1] + A[i-2]
```

TL;DR dynamic programming

This second way is *much* faster

It turns out you can take pretty much any recursion and solve it this way (called “dynamic programming”)

It can use a bit more memory,
but much faster

TL;DR dynamic programming

How many multiplication operations does it take to compute:

x^4?

x^{10}?

TL;DR dynamic programming

How many multiplication operations does it take to compute:

x^4? Answer: 2

x^{10}? Answer: 4

TL;DR dynamic programming

Can compute x^4 with 2 operations:

$x^2 = x * x$ (store this value)

$x^4 = x^2 * x^2$

Save CPU by using more memory!

Can compute x^n using $O(\lg n)$ ops

Also true if x is a matrix

Shortest paths using matrices

Any sub-path ($p_{x,y}$) of a shortest path ($p_{u,v}$) is also a shortest path

Thus we can recursively define a shortest path $p_{0,k} = \langle v_0, \ldots, v_k \rangle$, as:

$$w(p_{0,k}) = \min_{\text{“}k-1\text{”}}(w(p_{0,\text{“}k-1\text{”}}) + w(\text{“}k-1\text{”}, k))$$

Shortest paths using matrices

Thus a shortest path (using less than m edges) can be defined as:

$L^m = l^m_{i,j} = \min_k(l^{m-1}_{i,k} + l^1_{k,j})$,
where L^1 is the edge weights matrix

Can use dynamic programming to find an efficient solution

Shortest paths using matrices

L^m is not the m^{th} power of L, but the operations are very similar:

$L^m = l^m_{i,j} = \min_k(l^{m-1}_{i,k} + l^1_{k,j})$ // ours

$L^m = l^m_{i,j} = \sum_k(l^{m-1}_{i,k} * l^1_{k,j})$ //real times

Thus we can use our multiplication saving technique here too!

(see: MatrixAPSPmult.java)

Shortest paths using matrices

```
All-pairs-shortest-paths(W)
L(1) = W, n = W.rows, m  = 1
while m < n
   L(2m) = ESP(L(m), L(m))
   m = 2m
return L(m)
```

(ESP is L min op on previous slide)

Shortest paths using matrices

Runtime:

$|V|^3 \lg |V|$

Correctness:

By definition (brute force with some computation savers)

Floyd-Warshall

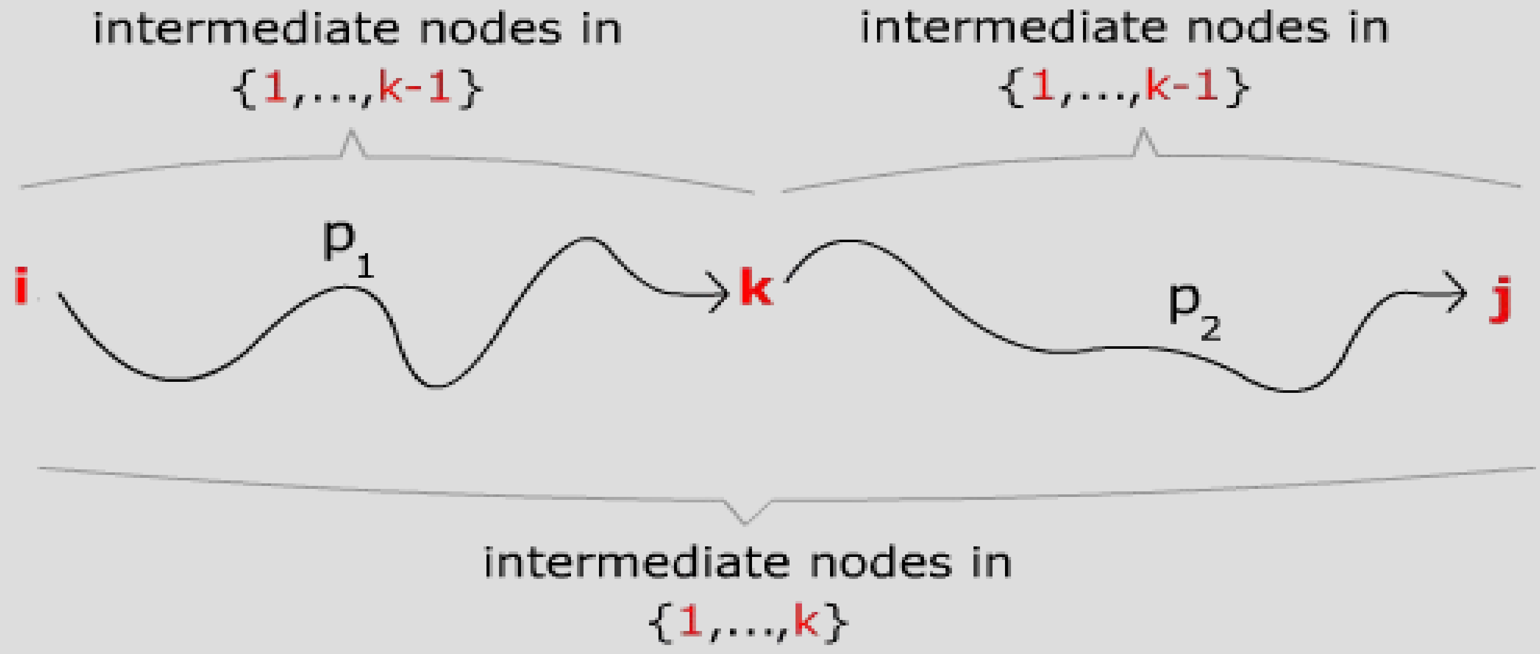
The Floyd-Warshall is similar but uses another shortest path property

Suppose we have a graph G, if we add a single vertex k to get G'

We now need to recompute all shortest paths

Floyd-Warshall

Either the path goes through k, or remains unchanged



$$d^{k}_{i,j} = \min\,(d^{k-1}_{i,j},\ d^{k-1}_{i,k} + d^{k-1}_{k,j})$$

Floyd-Warshall

Floyd-Warshall(W) // dynamic prog

$d^0_{i,j} = W_{i,j}$, $n = W.rows$

for $k = 1$ to n

  for $i = 1$ to n

    for $j = 1$ to n

      $d^k_{i,j} = \min(d^{k-1}_{i,j}, d^{k-1}_{i,k} + d^{k-1}_{k,j})$

Floyd-Warshall

Runtime:
$O(|V|^3)$

Correctness:
Again, by definition of shortest path