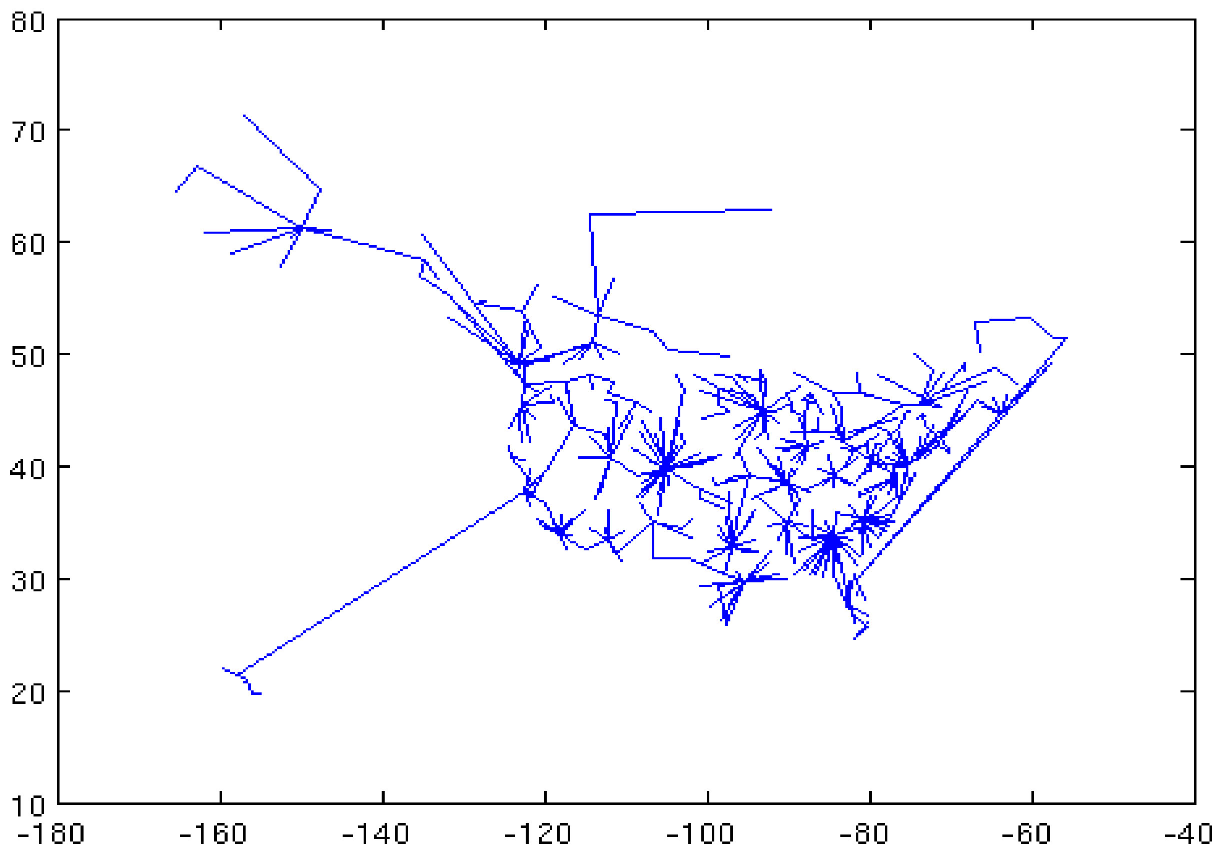
Minimum Spanning Tree (undirected graph)



Path tree vs. spanning tree

We have constructed trees in graphs for shortest path to anywhere else (from vertex is the root)

Minimum spanning trees instead want to connect every node with the least cost (undirected edges)

Path tree vs. spanning tree

Example: build the least costly road that allows cars to get from any start to any finish

Safe edges

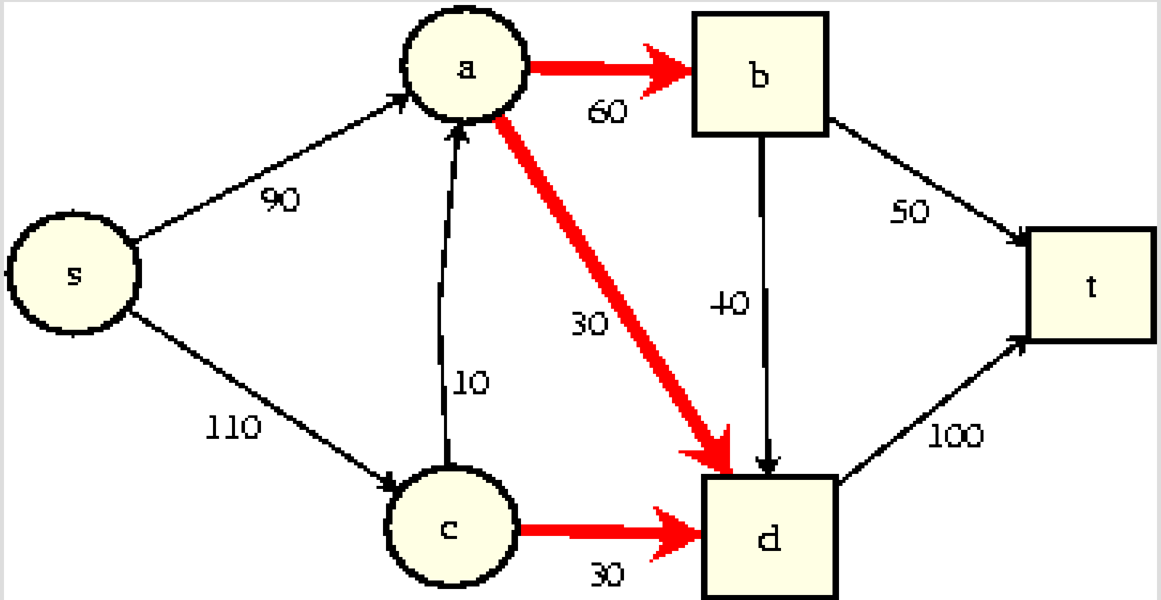
We an find (again) a greedy algorithm to solve MSTs

We can repeatedly add safe edges to an existing solution:

1. Find (u,v) as safe edge for *A*
2. Add (u,v) to *A* and repeat 1.

Safe edges

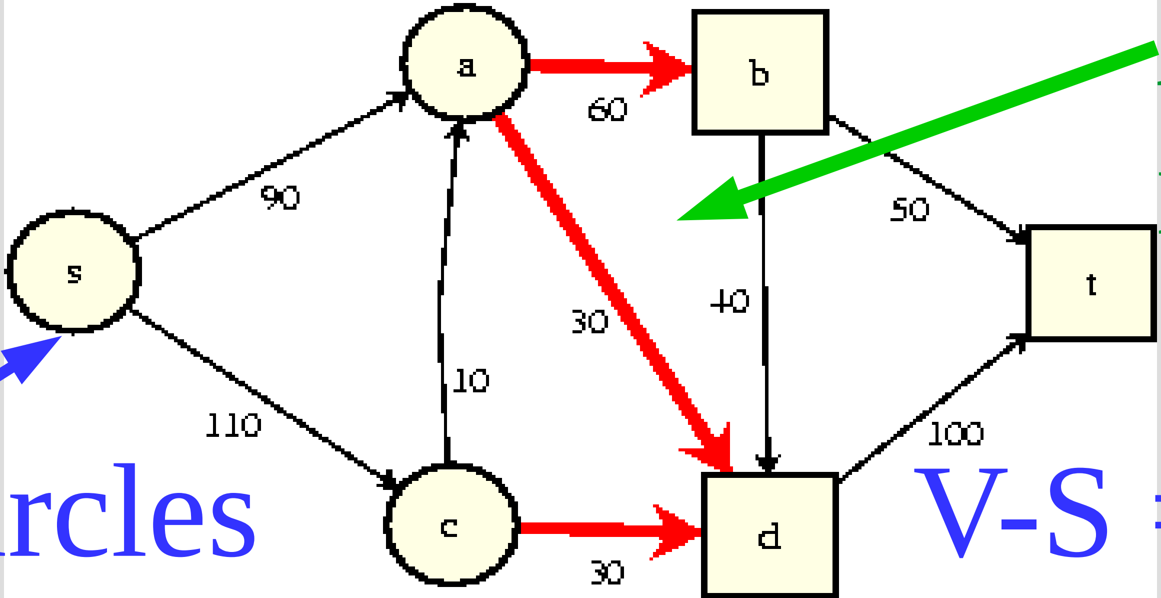
A <u>cut</u> S: (S, V-S) for any verticies S
Cut S <u>respects</u> *A*: no edge in *A* has one side in S and another in V-S



Safe edges

A cut S: (S, V-S) for any verticies S
Cut S respects A: no edge in A has one side in S and another in V-S

S respects A if no red edges



S = circles

V-S = squares

Safe edges

Theorem 23.1:
Let A be a set of edges that is included in some MST

Let S be a cut that respects A

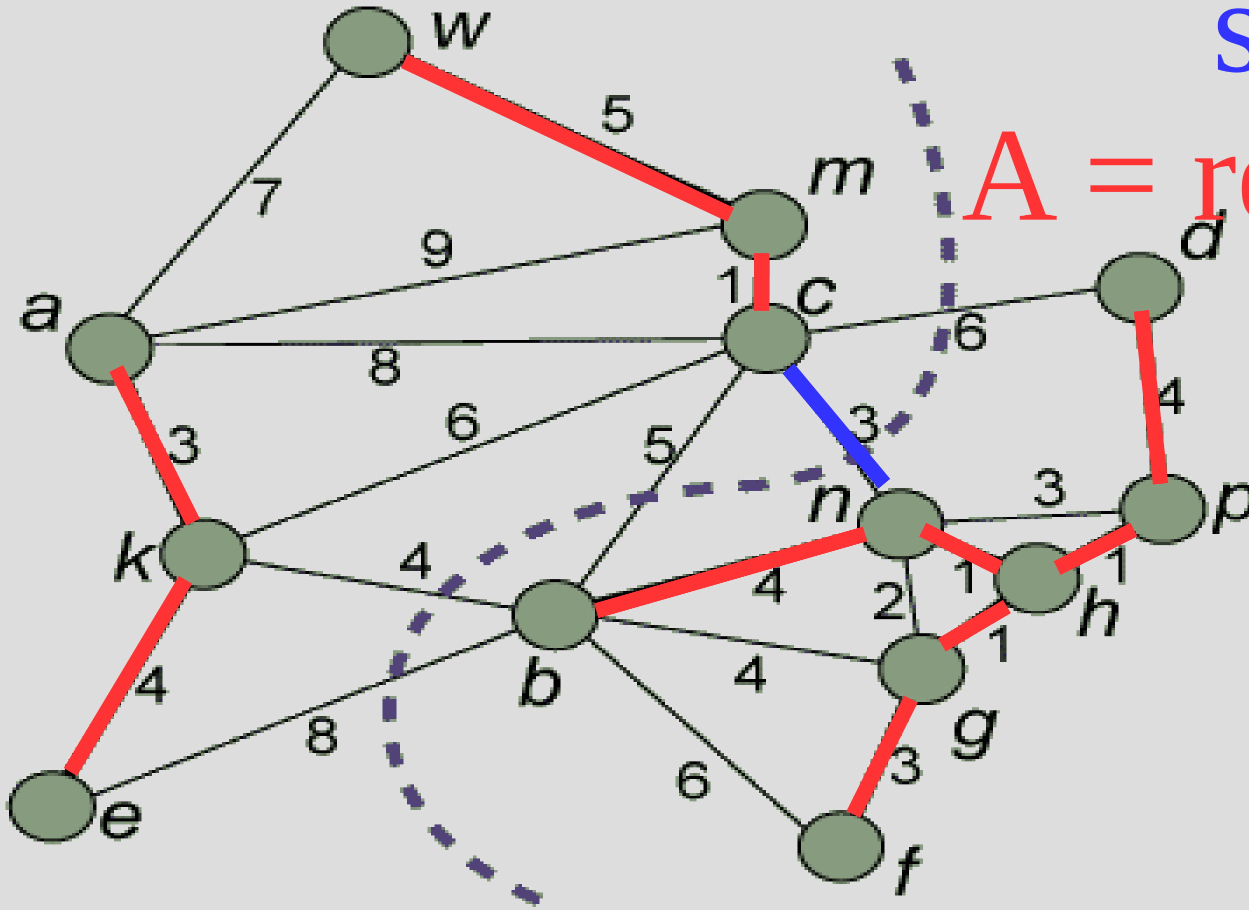
Then the minimum edge that crosses S and V-S is a safe edge for A

Safe edges

Theorem 23.1:

blue = minimum safe edge

A = red edges



LHS = S

RHS = V-S

Safe edges

Proof:

Let T be a MST that includes A

Add minimum safe edge (u,v)

Let (x,y) be the other edge on the cut

Remove (x,y), and call this T' thus:

$w(T') = w(T) + w(u,v) - w(x,y)$

But (u,v) min, so $w(u,v) \leq w(x,y)$

Thus, $w(T') \leq w(T)$ and we done

Kruskal

Idea:

1. Sort all edges into a list

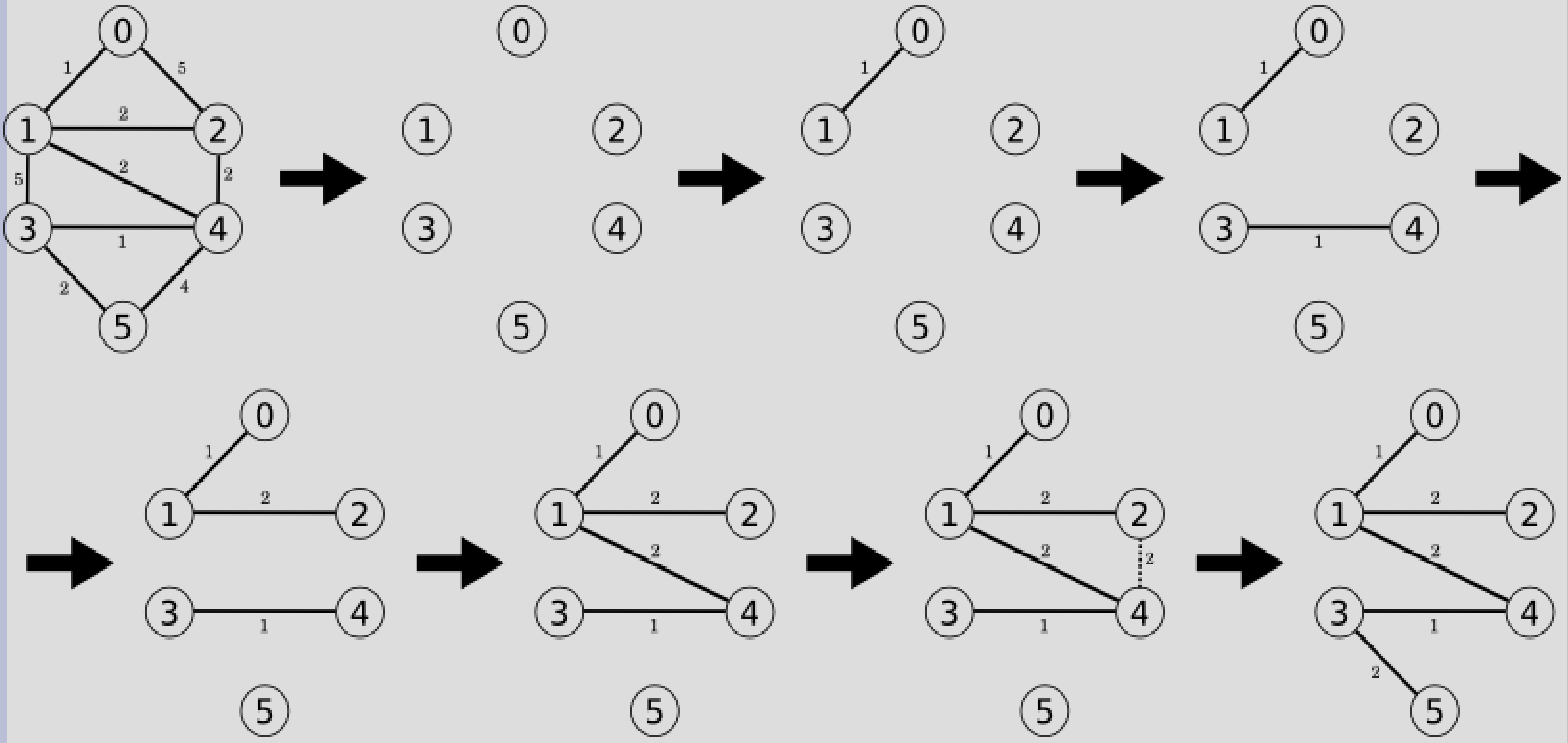
2. If the minimum edge in the list does not create a cycle, add it to A

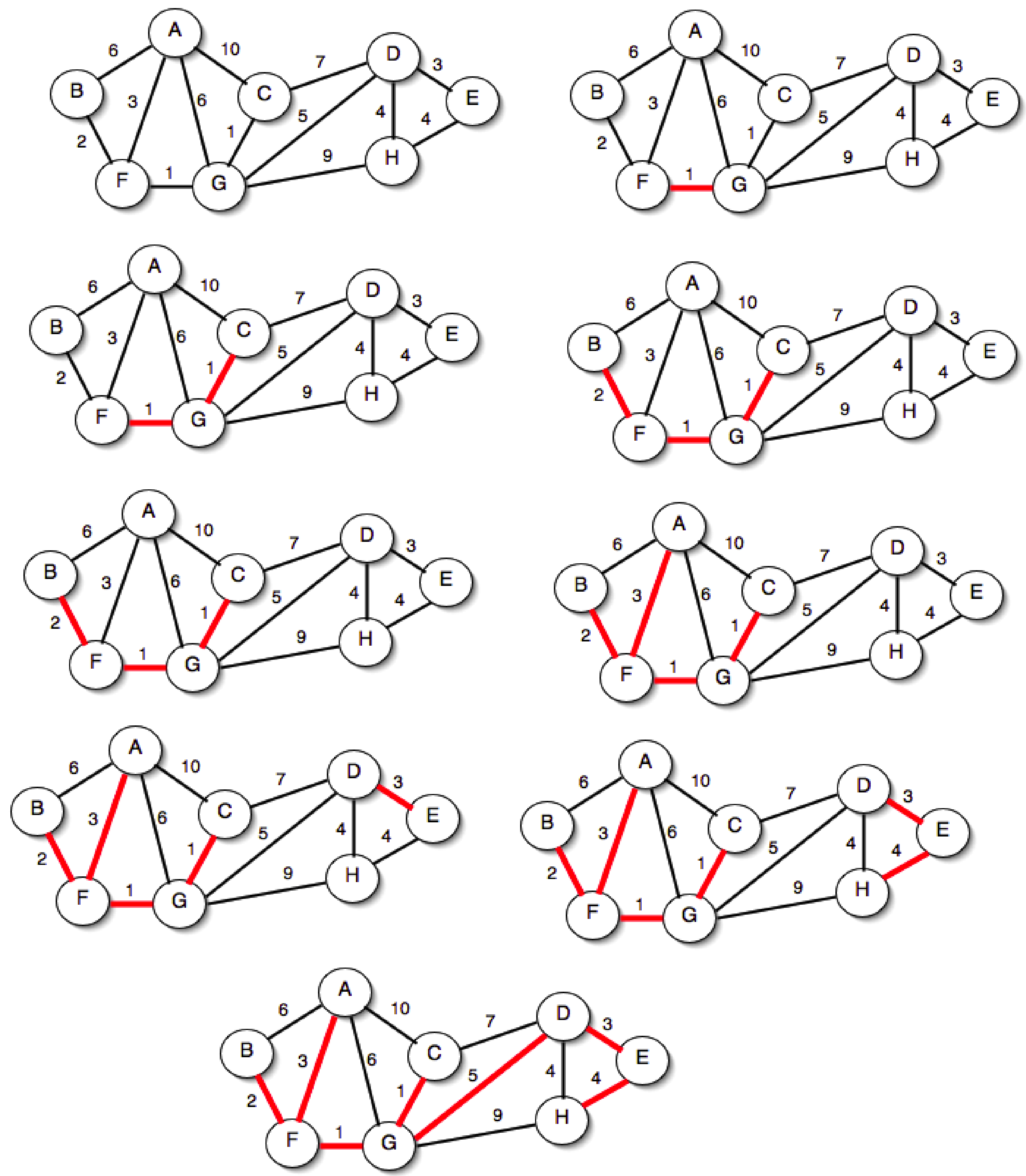
3. Remove the edge and repeat 2 until no more edges

Kruskal

```
MST-Kruskal(G,w)
A = { }
for each v in G.V: Make-Set(V)
sort(G.E)
for (u,v) in G.E (w(u,v) increasing)
   if Find-Set(u) ≠ Find-Set(v)
      A= A U {(u,v)}
      Union(u,v)
```

Kruskal




A
B
C
D
E
F
G
H
6
10
7
3
3
6
1
5
4
4
2
1
9

Kruskal

Runtime:

Find-Set takes about $O(\lg |V|)$ time (Ch. 21)

Thus overall is about $O(|E| \lg |V|)$

Prim

Idea:

1. Select any vertex (as the root)
2. Find the shortest edge from a vertex in the tree to a vertex outside
3. Add this edge (and the connected vertex) to the tree
4. Goto 2.

Like Dijkstra, but different relaxation

Prim

```
MST-Prim(G, w, r) // r is root
for each u in G.V: u.key=∞, u.π=NIL
r.key = 0, Q = G.V
while Q not empty
   u = Extract-Min(Q)
   for each v in G.Adj[u]
      if v in Q and w(u,v) < v.key
         v.key=w(u,v), v.π=u
```

modified "relax" from Dijkstra

Prim

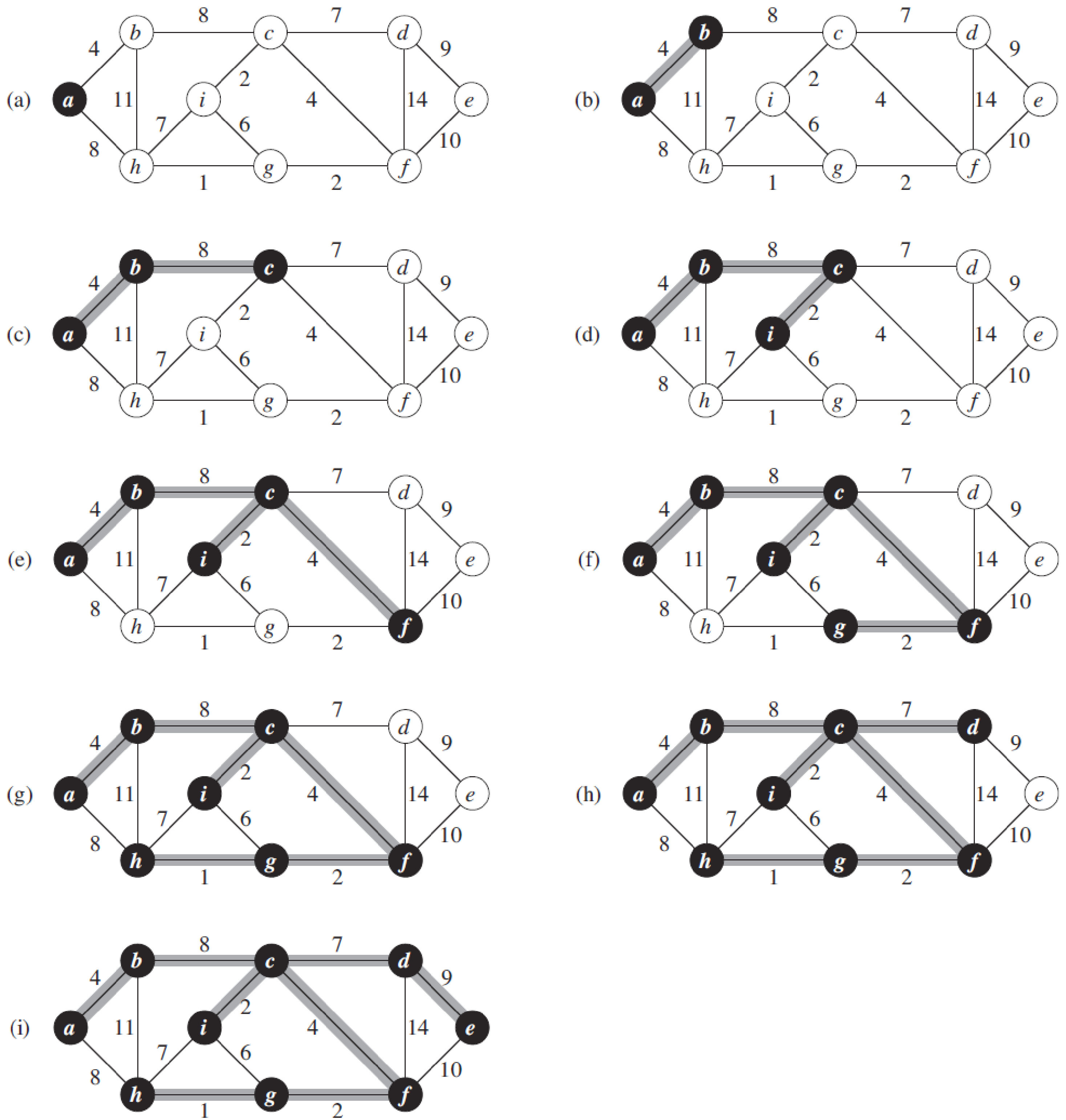
Runtime:

Extract-Min(V) is $O(\lg |V|)$, run $|V|$ times is $O(|V| \lg |V|)$

for loop runs over each edge twice, minimizing (i.e. Decrease-Key())...

$O((|V|+|E|) \lg |V|) = O(|E| \lg |V|)$

(Fibonacci heaps $O(|E| + |V| \lg |V|)$)


(a)
(b)
(c)
(d)
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(f)
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(h)
(i)