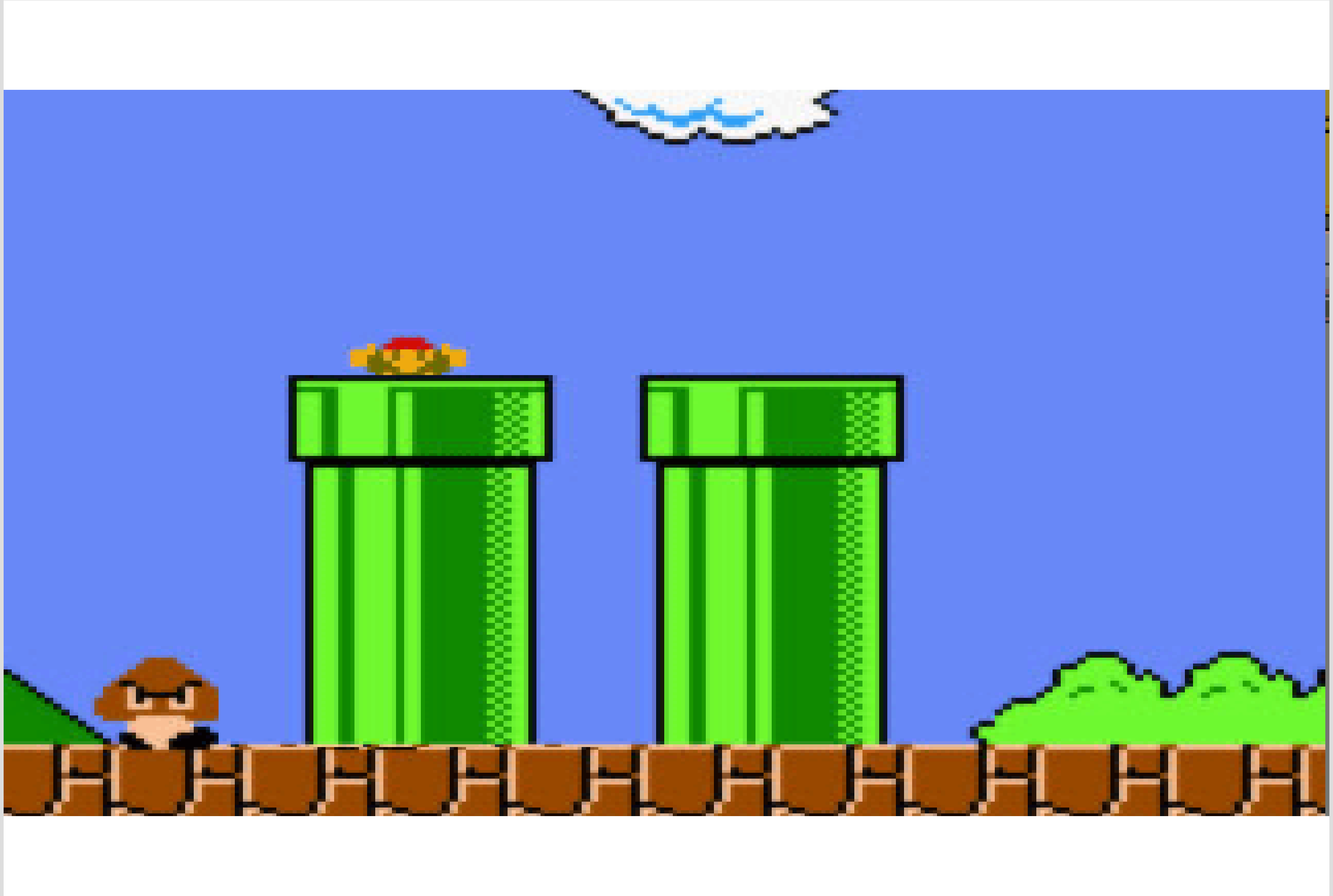
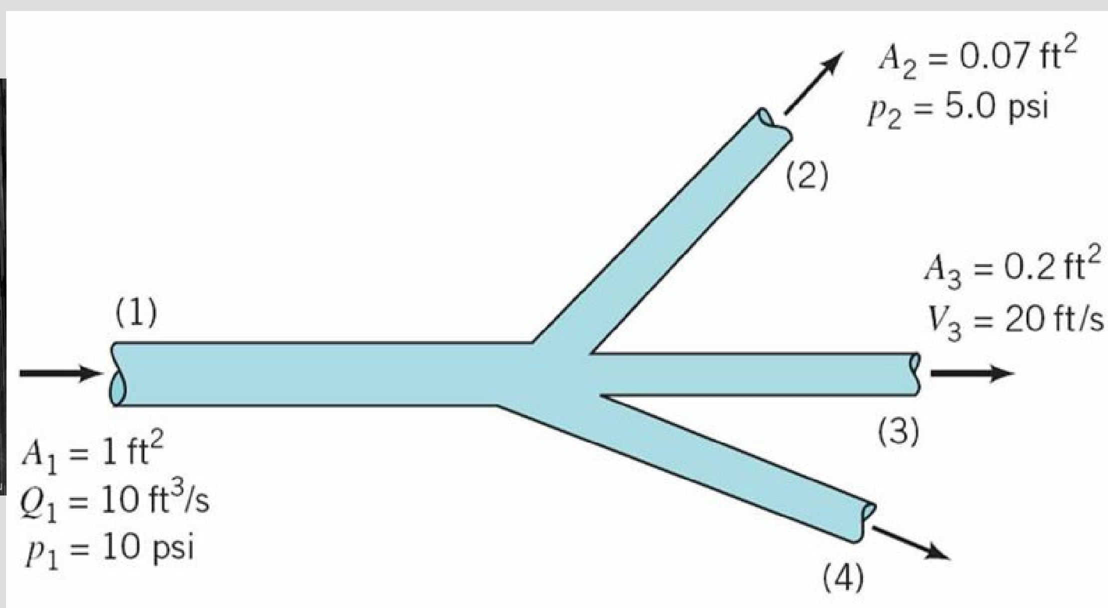
Network Flow



Network Flow terminology

Network flow is similar to finding how much water we can bring from a “source” to a “sink” (infinite) (intermediates cannot “hold” water)



Network Flow terminology

Definitions:

$c(u,v)$: edge <u>capacity</u>, $c(u,v) \geq 0$

$f(u,v)$: flow from u to v s.t.

1. $0 \leq f(u,v) \leq c(u,v)$
2. $\sum_v f(u,v) = \sum_v f(v,u)$

flow out
=flow in

s : a <u>source</u>, $\sum_v f(s,v) \geq \sum_v f(v,s)$

t : a <u>sink</u>, $\sum_v f(t,v) \leq \sum_v f(v,t)$

Network Flow terminology

Definitions (part 2):

$|f| = \sum_v f(s,v) - \sum_v f(v,s)$

^ amount of flow from source

Want to maximize $|f|$ for the <u>maximum-flow problem</u>

Network Flow terminology

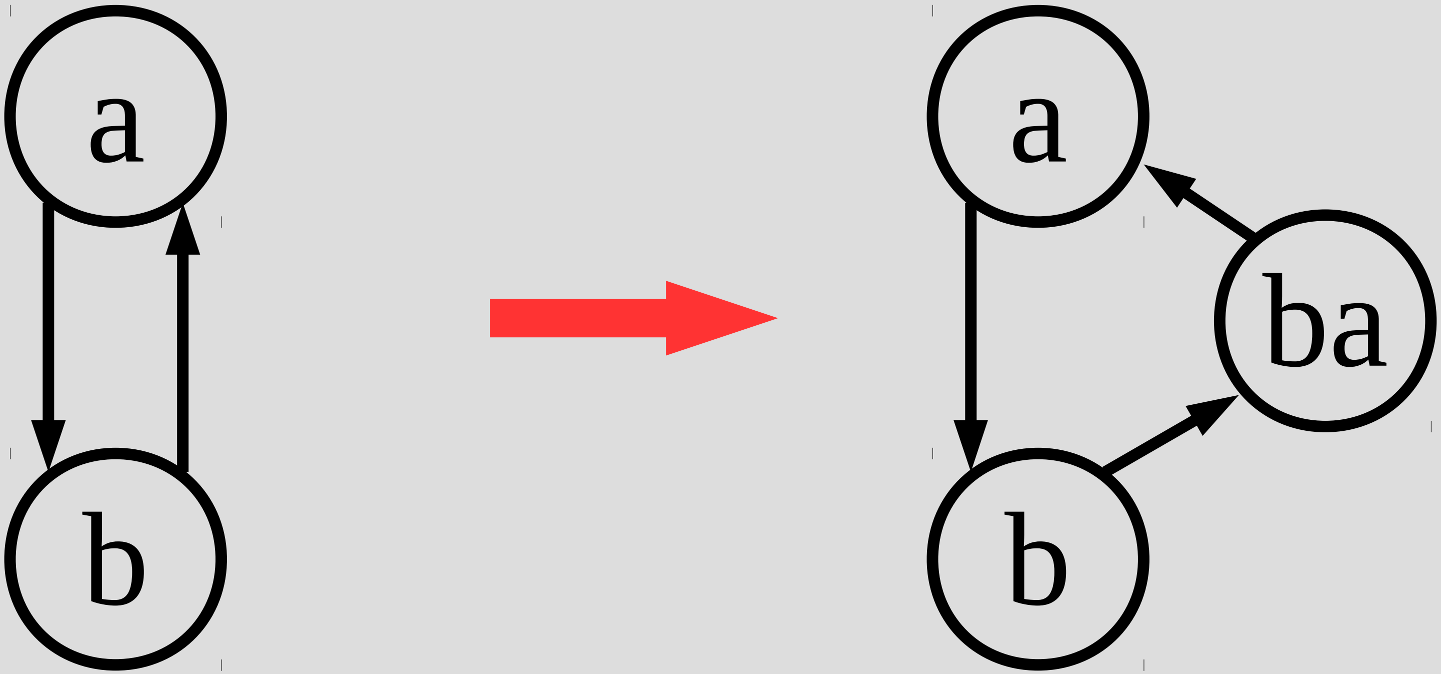
Graph restrictions:

1. If there is an edge (u,v), then there cannot be edge (v,u)
2. Every edge is on a path from source to sink
3. One sink and one source

(None are really restrictions)

Network Flow terminology

1. If there is an edge (u,v), then there cannot be edge (v,u)

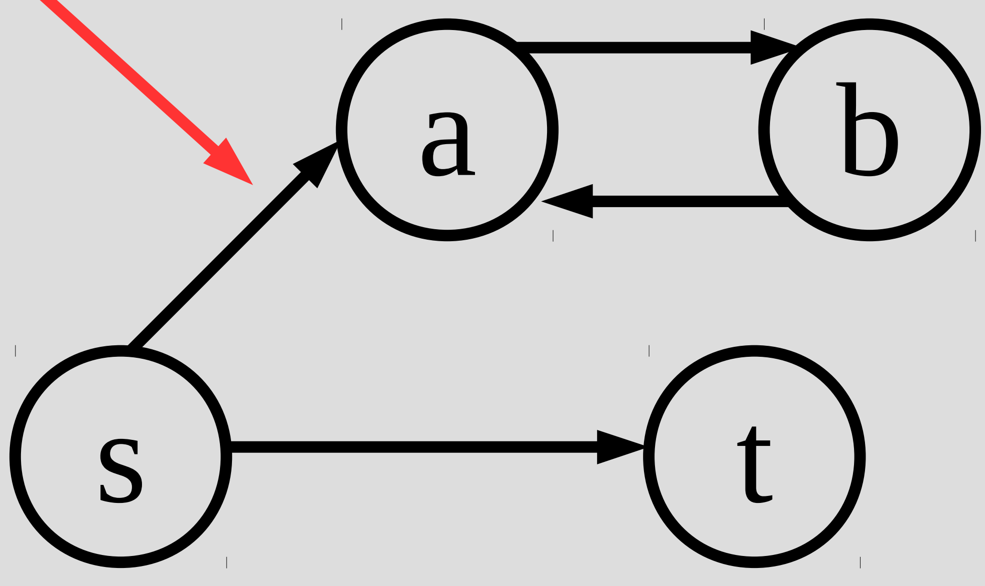


Network Flow terminology

2. Every edge is on a path from source to sink

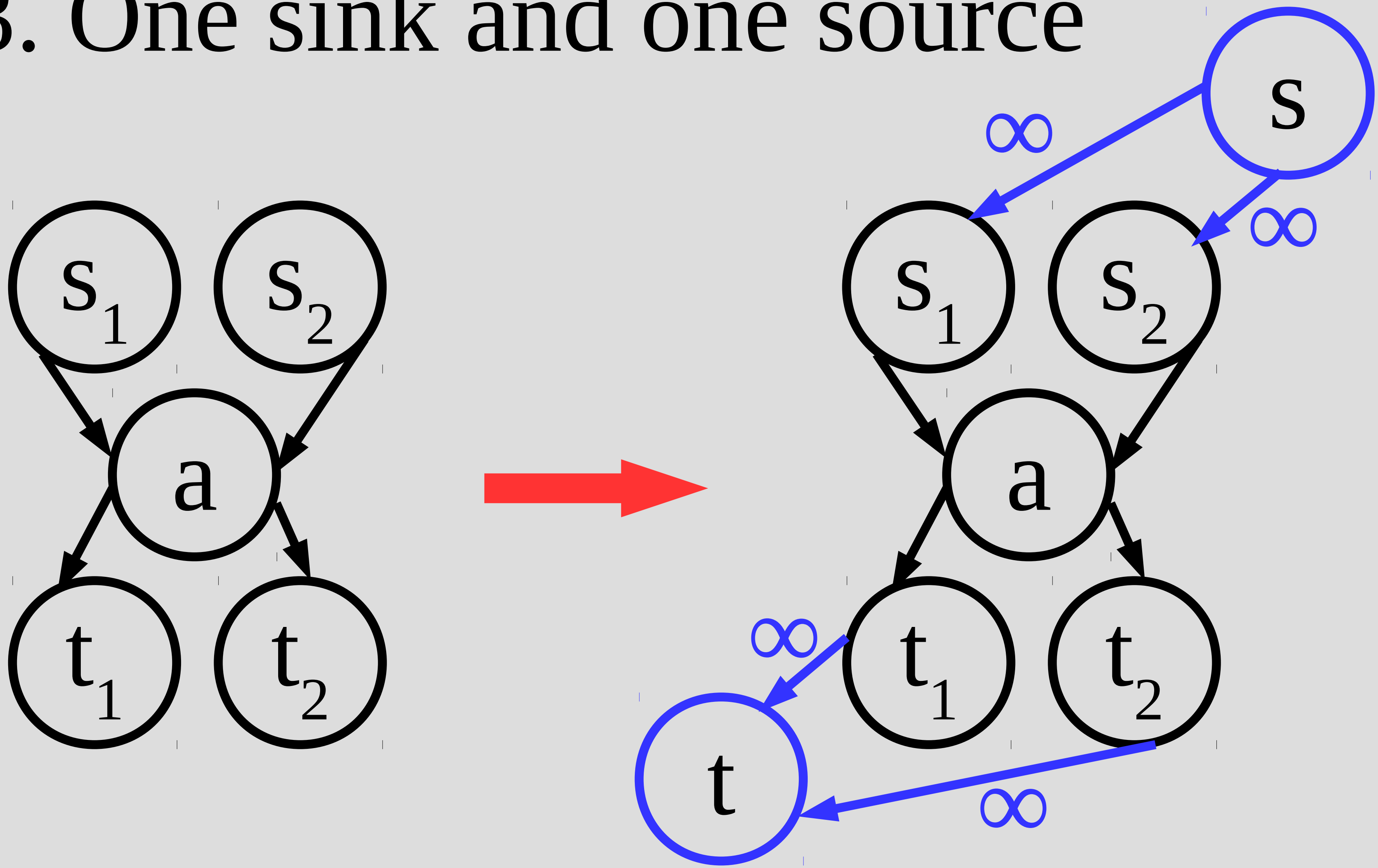
flow in = flow out,
only possible
flow in is 0

(worthless edge)



Network Flow terminology

3. One sink and one source



Ford-Fulkerson

Idea:

1. Find a path from source to sink
2. Add maximum flow along path (minimum capacity on path)
3. Repeat

Note: this path needs to be found in a "residual" graph

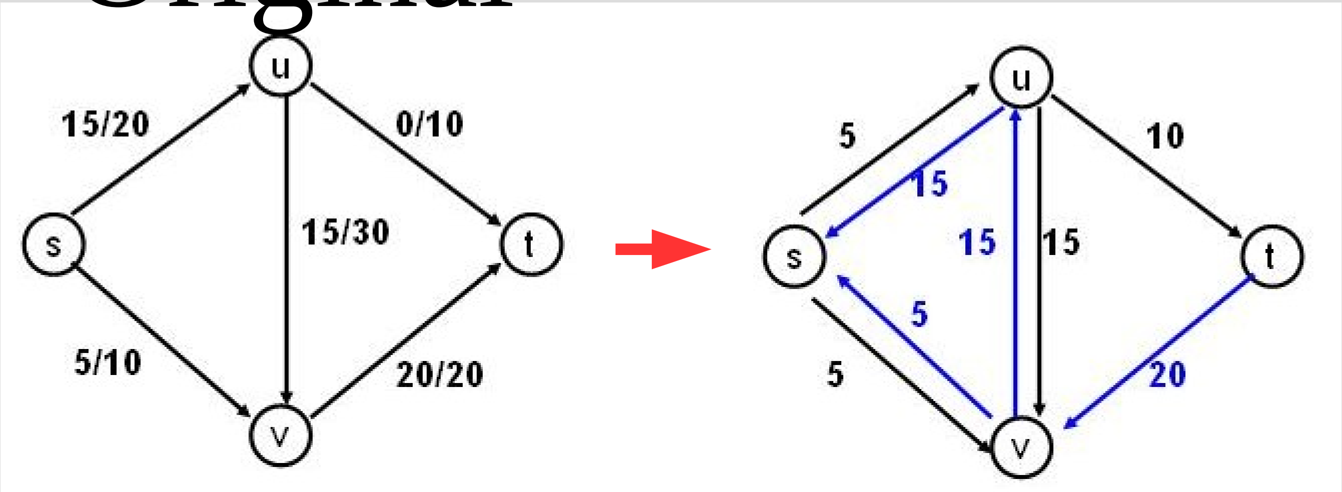
Ford-Fulkerson

What is a residual graph?

Forward edges = capacity left

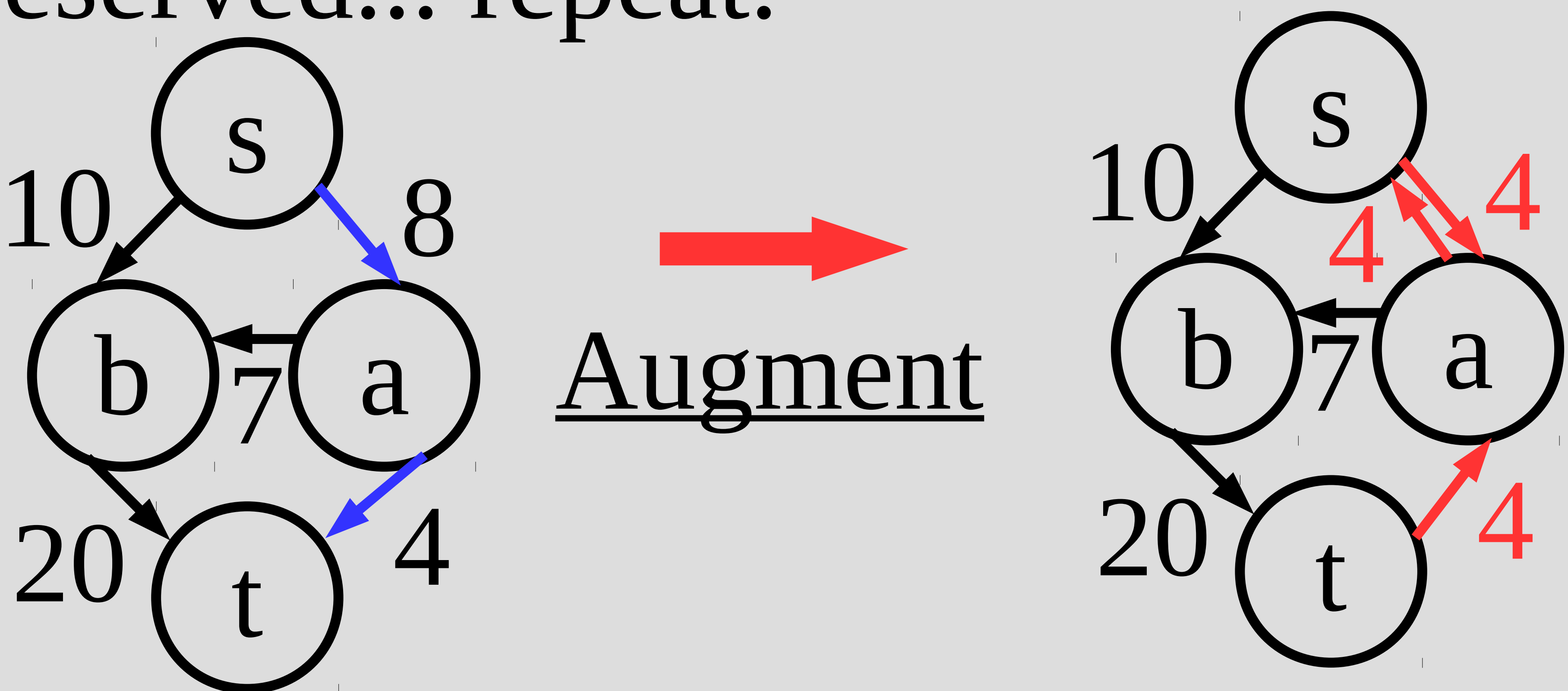
Back edges = flow

Original Residual



Ford-Fulkerson

Idea: Find a way to add some flow, modify graph to show this flow reserved... repeat.

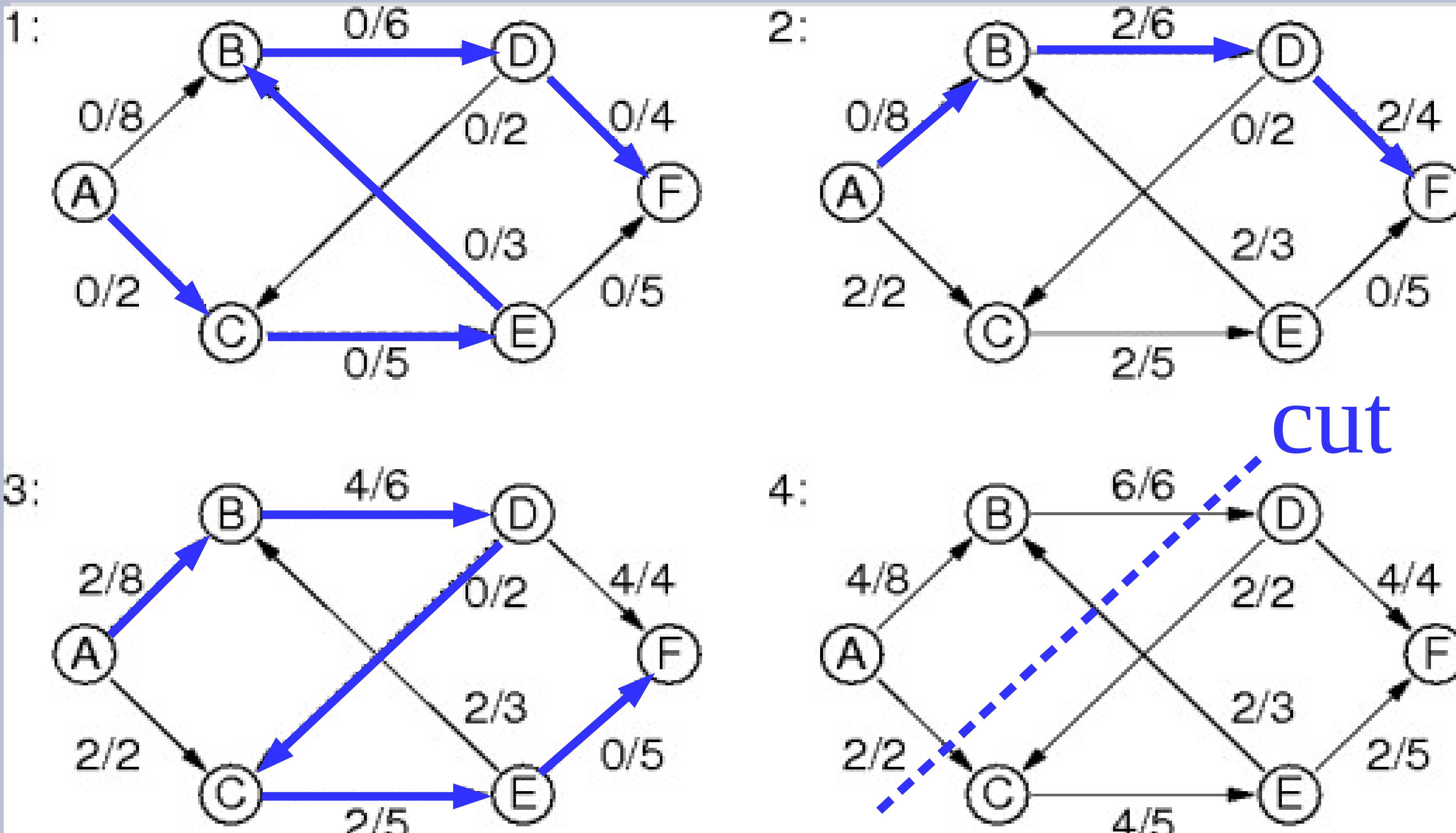


Ford-Fulkerson

Ford-Fulkerson(G, s, t)
initialize network flow to 0
while (exists path from s to t)
   augment flow, f, in G along path
return f

(Note: “augment flow” means add this flow to network)

Ford-Fulkerson



Ford-Fulkerson

Subscript "f" denotes <u>residual</u> (or modified graph)

G_f = residual graph

E_f = residual edges

c_f = residual capacity

$c_f(u,v) = c(u,v) - f(u,v)$

$c_f(v,u) = f(v,u)$

"forward edge"
capacity - flow

"back edge"
just flow

Ford-Fulkerson

Ford-Fulkerson(G, s, t)
for: each edge (u,v) in G.E: (u,v).f=0
while: exists path from s to t in G_f
  find c_f(p) // minimum edge cap. on path
  for: each edge (u,v) in p
    if(u,v) in E: (u,v).f=(u,v).f + c_f(p)
    else: (u,v).f=(u,v).f - c_f(p)

Ford-Fulkerson

Runtime:

How hard is it to find a path?

How many possible paths could you find?

Ford-Fulkerson

Runtime:

How hard is it to find a path?
-$O(E)$ (via BFS or DFS)
How many possible paths could you find?
- $|f^*|$ (paths might use only 1 flow)
.... so, $O(E\,|f^*|)$

Ford-Fulkerson

$(f \uparrow f')(u,v)$ = flow f augmented by f'

$(f \uparrow f')(u,v) = f(u,v) + f'(u,v) - f'(v,u)$

Lemma 26.1: Let f be the flow in G, and f' be a flow in G_f, then $(f \uparrow f')$ is a flow in G with total amount:

$|f \uparrow f'| = |f| + |f'|$

Proof: pages 718-719

Ford-Fulkerson

For some path p:

$c_f(p) = \min(c_f(u,v) : (u,v) \text{ on } p)$

^^ (capacity of path is smallest edge)

Claim 26.3:

Let $f_p = c_f(p)$, then

$|f \uparrow f_p| = |f| + |f_p|$

Ford-Fulkerson

More bad notation:

c(u,v) = capacity of an edge
if u and v are single vertexes

c(S,T) = capacity across a cut
if S and T are sets of vertexes

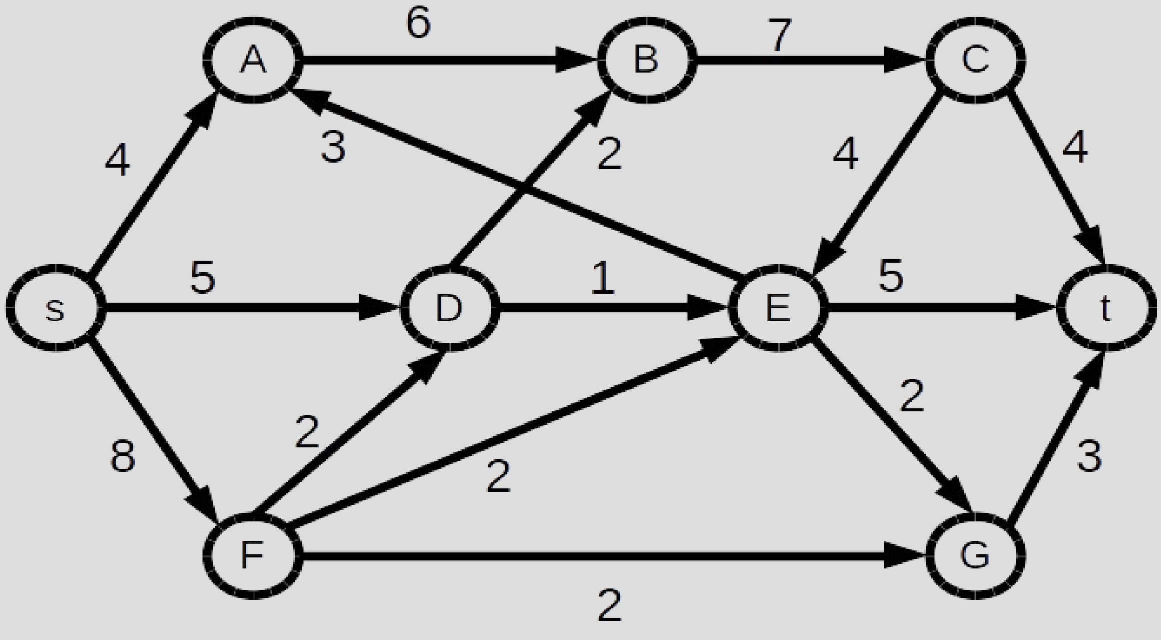
... Similarly for f(u,v) and f(S,T)

Max flow, min cut

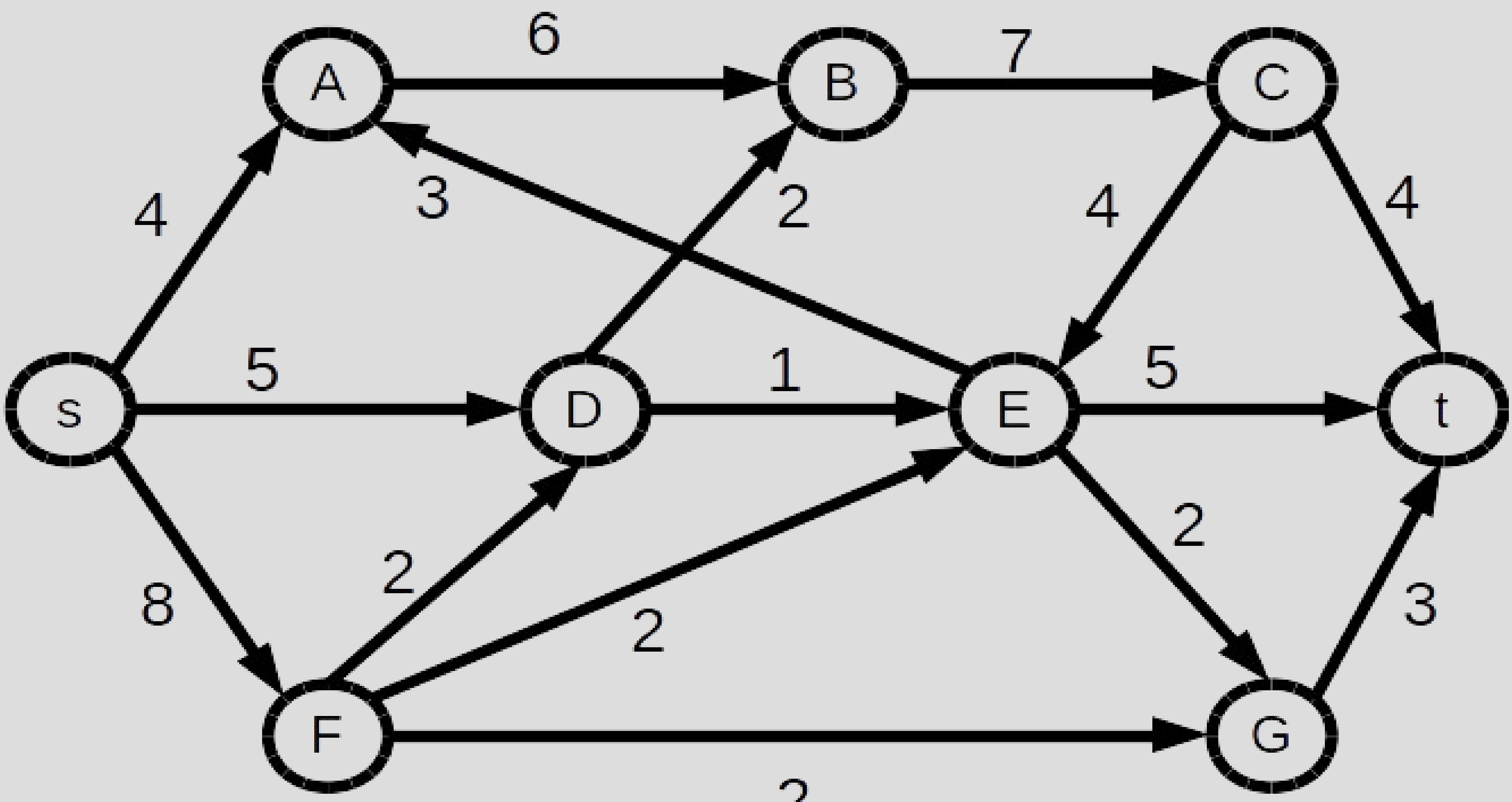
Relationship between cuts and flows?

$$c(S,T) = \sum_{u \text{ in } S}\sum_{v \text{ in } T} c(u,v)$$

$$f(S,T) = \sum_{u \text{ in } S}\sum_{v \text{ in } T} f(u,v)-\sum_{u}\sum_{v} f(v,u)$$



Max flow, min cut

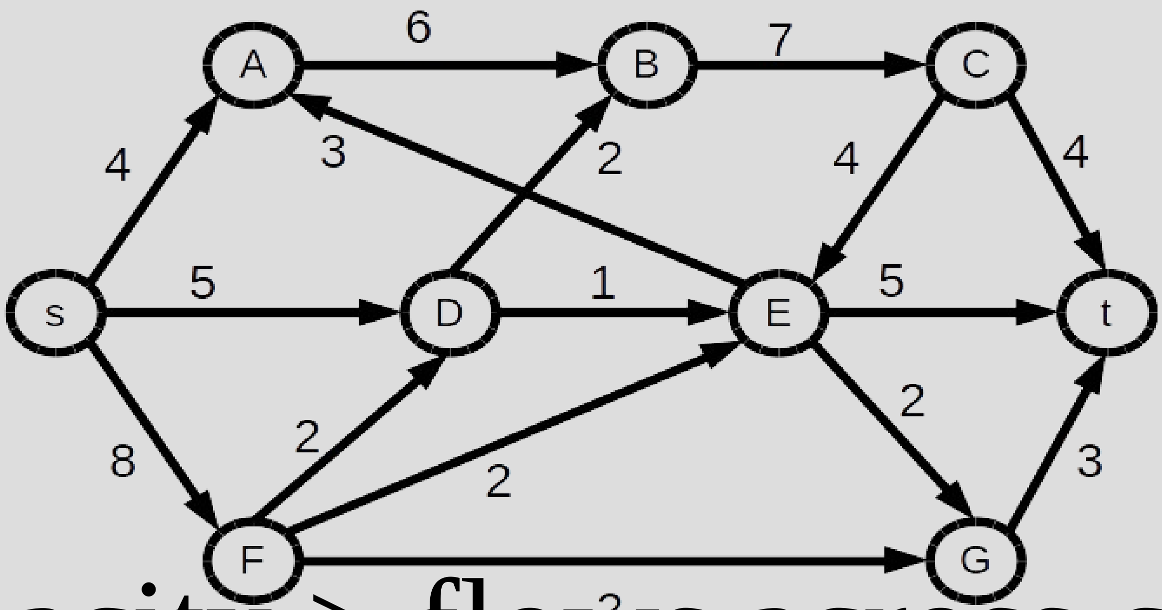


Max flow, min cut

Relationship between cuts and flows?

$c(S,T) = \sum_{u \text{ in } S}\sum_{v \text{ in } T} c(u,v)$

$f(S,T) = \sum_{u \text{ in } S}\sum_{v \text{ in } T} f(u,v)-\sum_{u}\sum_{v} f(v,u)$



cut capacity $\geq$ flows across cut

Max flow, min cut

Lemma 26.4

Let (S,T) be any cut, then f(S,T) = |f|

Proof:

Page 722

(Kinda long)

Max flow, min cut

Corollary 26.5

Flow is not larger than cut capacity

Proof:

$$
\begin{aligned}
|f| &= \sum_{u \text{ in } S}\sum_{v \text{ in } T} f(u,v) - \sum_u \sum_v f(v,u) \\
&\le \sum_{u \text{ in } S}\sum_{v \text{ in } T} f(u,v) \\
&\le \sum_{u \text{ in } S}\sum_{v \text{ in } T} c(u,v) \\
&= c(S,T)
\end{aligned}
$$

Max flow, min cut

Theorem 26.5

All 3 are equivalent:

1. f is a max flow
2. Residual network has no aug. path
3. $|f| = c(S,T)$ for some cut (S,T)

maximum network flow

= min cut (i.e. bottlneck)

Proof:

Will show: 1 => 2, 2=>3, 3=>1

Max flow, min cut

f is a max flow => Residual network has no augmenting path

Proof:

Assume there is a path p

$|f \uparrow f_p| = |f| + |f_p| > |f|$, which is a contradiction to $|f|$ being a max flow

Max flow, min cut

Residual network has no aug. path => |f| = c(S,T) for some cut (S,T)

Proof:

Let S = all vertices reachable from s in G_f

u in S, v in T => f(u,v) = c(u,v) else there would be path in G_f

Max flow, min cut

Also, $f(v,u) = 0$ else $c_f(u,v) > 0$ and again v would be reachable from s

$$
\begin{aligned}
f(S,T) &= \sum_{u \text{ in } S} \sum_{v \text{ in } T} f(u,v) - \sum_u \sum_v f(v,u) \\
&= \sum_{u \text{ in } S} \sum_{v \text{ in } T} c(u,v) - \sum_u \sum_v 0 \\
&= c(S,T)
\end{aligned}
$$

Max flow, min cut

$|f| = c(S,T)$ for some cut (S,T)
$\Rightarrow f$ is a max flow

Proof:
$|f| \leq c(S,T)$ for all cuts (S,T)

Thus trivially true, as $|f|$ cannot get larger than $C(S,T)$

Edmonds-Karp

exists shortest path (BFS)

~~Ford-Fulkerson~~(G, s, t)

for: each edge (u,v) in G.E: (u,v).f=0

while: ~~exists path from s to t in G_f~~

find c_f(p) // minimum edge cap.

for: each edge (u,v) in p

if(u,v) in E: (u,v).f=(u,v).f + c_f(p)

else: (u,v).f=(u,v).f - c_f(p)

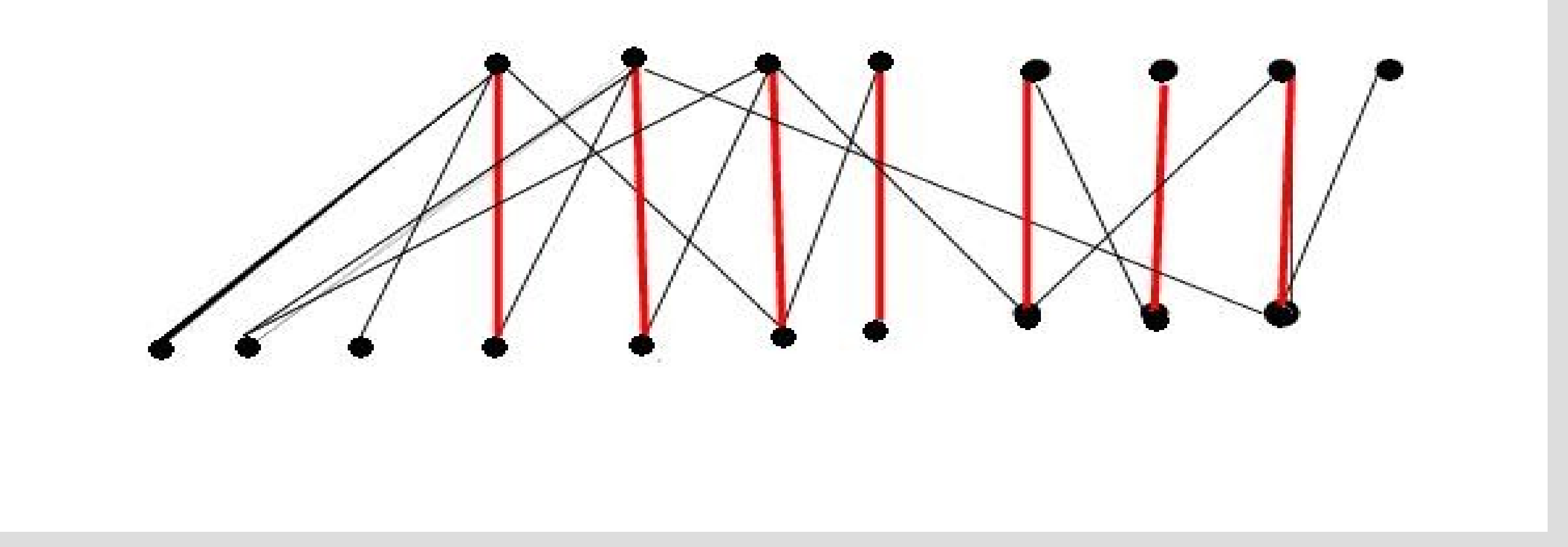
Edmonds-Karp

Lemma 26.7
Shortest path in G_f is non-decreasing

Theorem 26.8
Number of flow augmentations by Edmonds-Karp is $O(|V||E|)$
So, total running time: $O(|V||E|^2)$

Matching

Another application of network flow is maximizing (number of)matchings in a bipartite graph



Each node cannot be “used” twice

Matching

Add “super sink” and “super source”
(and direct edges source -> sink)
capacity = 1 on all edges