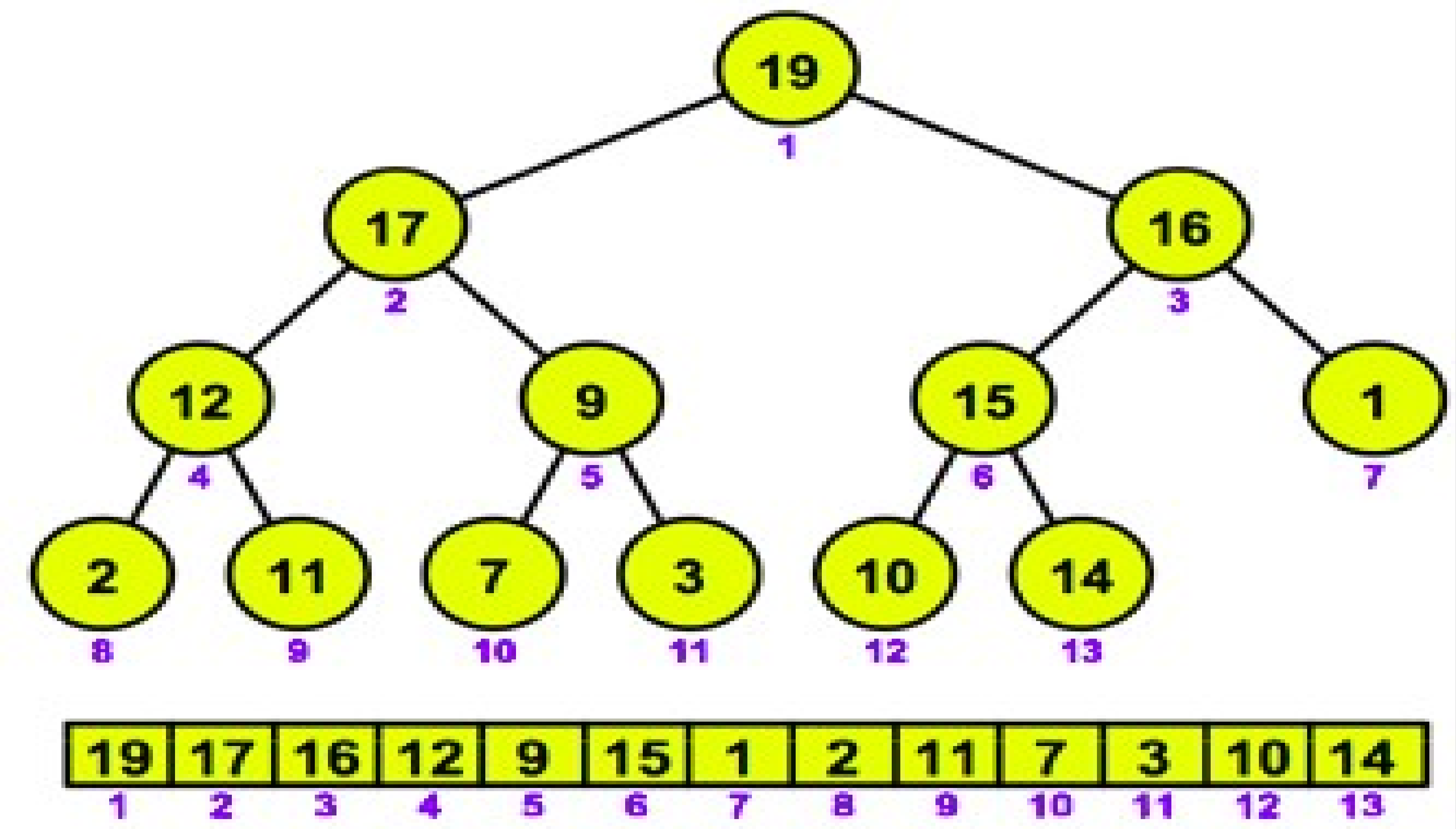
Heapsort


19
1
17
2
16
3
12
4
9
5
15
6
1
7
2
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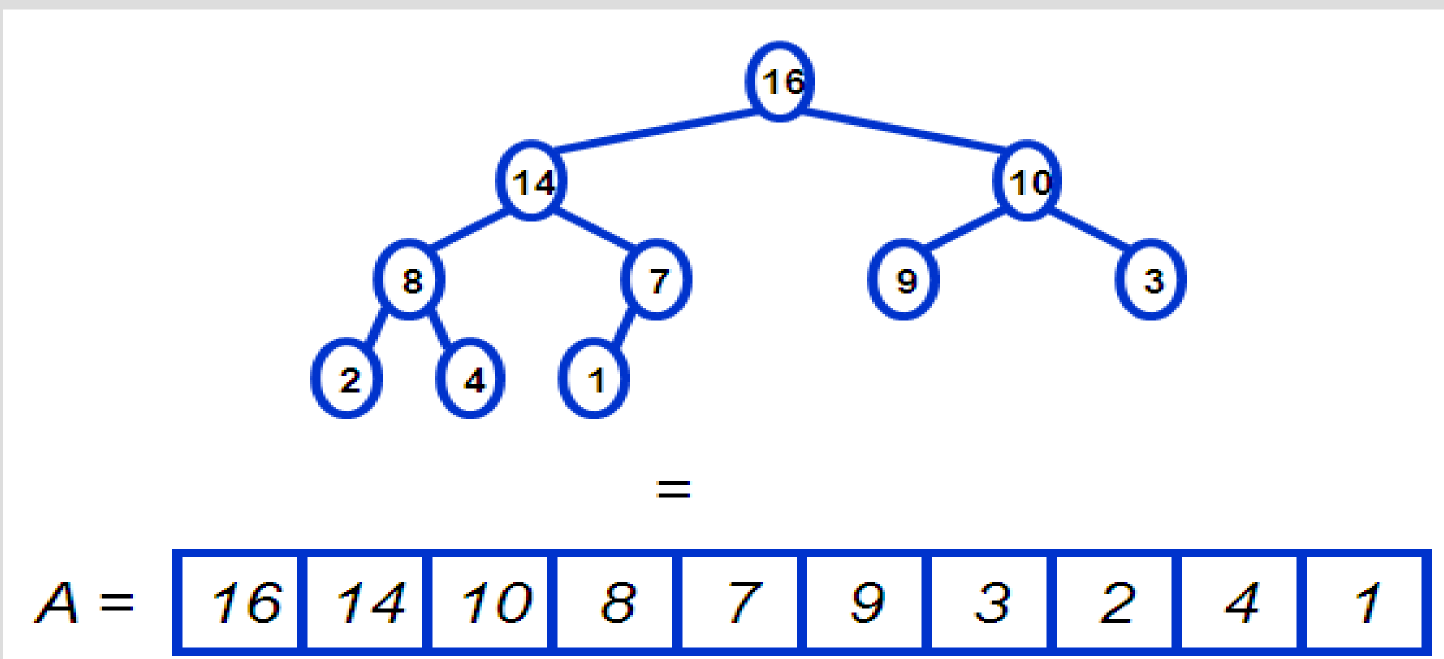
3
11
10
12
14
13
19 17 16 12 9 15 1 2 11 7 3 10 14
1 2 3 4 5 6 7 8 9 10 11 12 13

Announcements

Moodle had issues last night, homework due tonight at 11:55pm

Binary tree as array

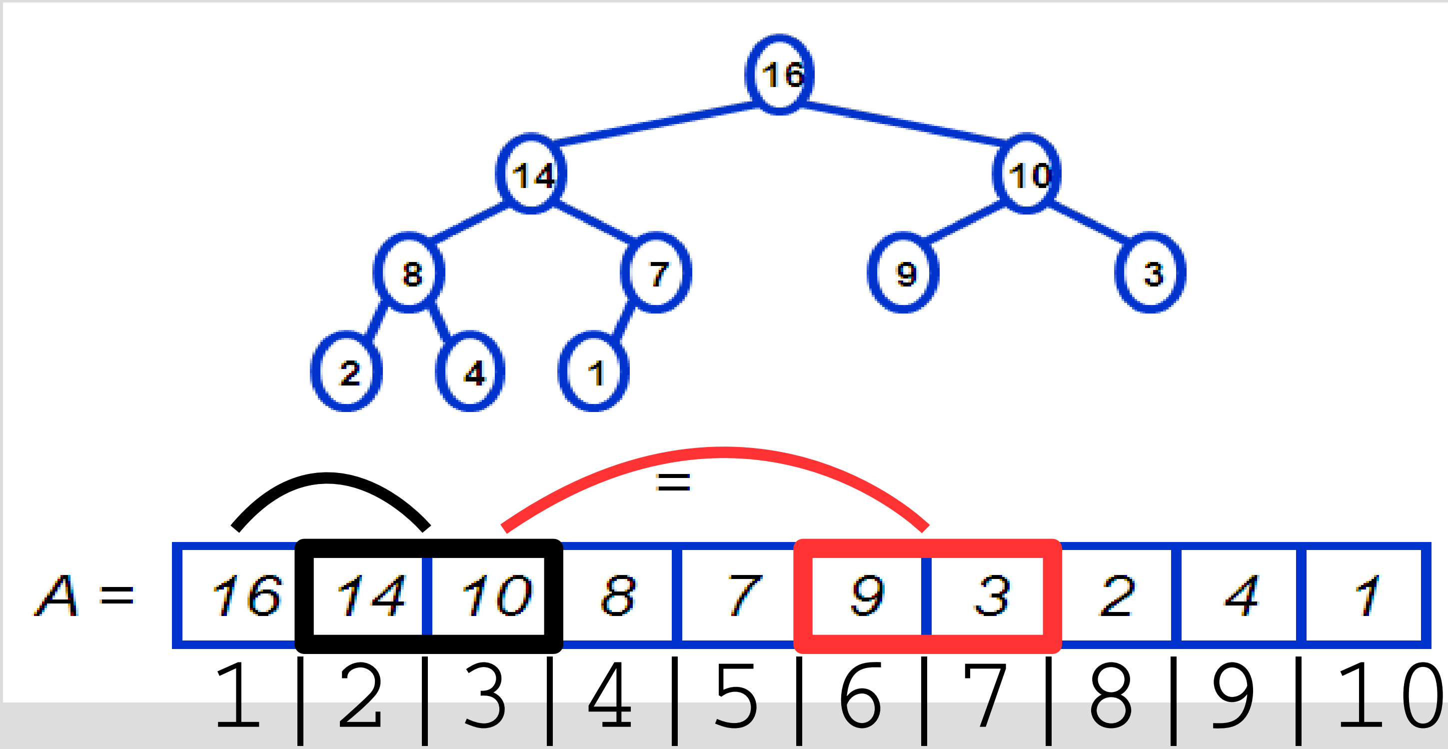
It is possible to represent binary trees as an array



1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10

Binary tree as array

index 'i' is the parent of '2i' and '2i+1'



Binary tree as array

Is it possible to represent any tree with a constant branching factor as an array?

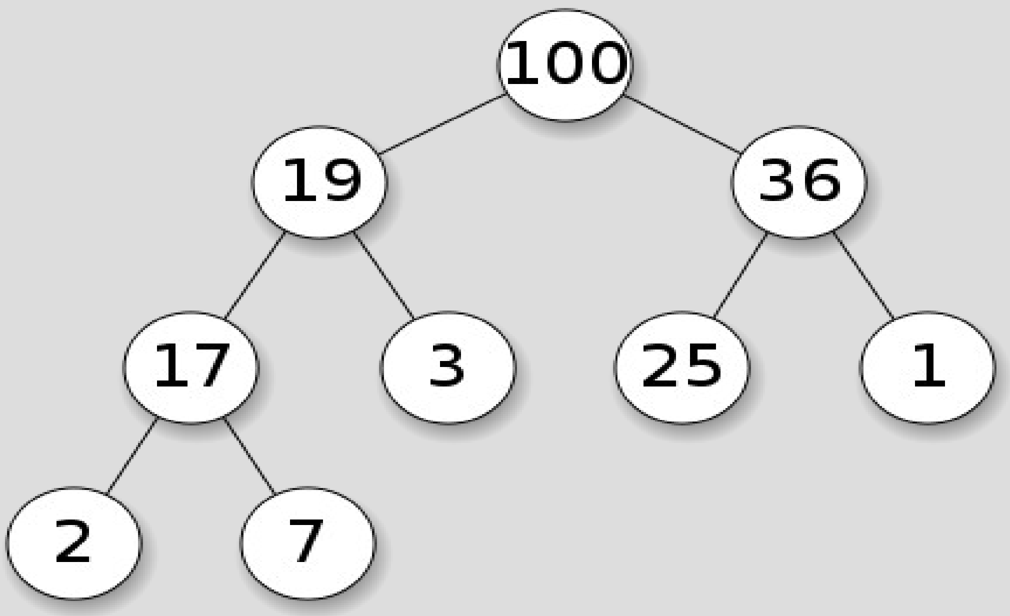
Binary tree as array

Is it possible to represent any tree with a constant branching factor as an array?

Yes, but the notation is awkward

Heaps

A max heap is a tree where the parent is larger than its children
(A min heap is the opposite)



Heapsort

The idea behind heapsort is to:

1. Build a heap
2. Pull out the largest (root) and re-compile the heap
3. (repeat)

Heapsort

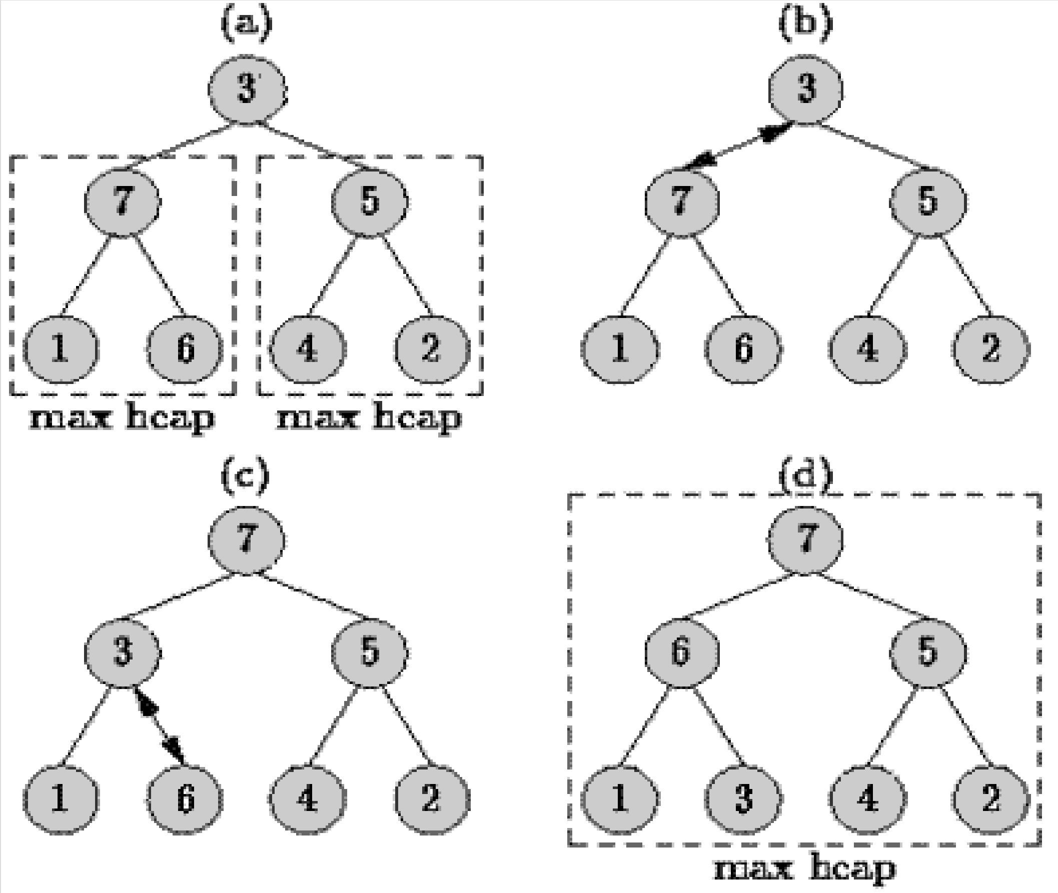
To do this, we will define subroutines:

1. Max-Heapify = maintains heap property

2. Build-Max-Heap = make sequence into a max-heap

Max-Heapify

Input: a root of two max-heaps
Output: a max-heap



Max-Heapify

```
Pseudo-code Max-Heapify(A,i):
left = left(i)      // 2*i
right = right(i)  // 2*i+1
L = arg_max( A[left], A[right], A[ i ])
if (L not i)
   exchange A[ i ] with A[ L ]
   Max-Heapify(A, L)
// now make me do it!
```

Max-Heapify

Runtime?

Max-Heapify

Runtime?

Obviously (is it?): lg n

T(n) = T(2/3 n) + O(1) // why?

Or...

T(n) = T(1/2 n) + O(1)

Master's theorem

Master's theorem: (proof 4.6)

For $a \geq 1$, $b \geq 1$, $T(n) = a\,T(n/b) + f(n)$

If f(n) is... (3 cases)

$O(n^c)$ for $c < \log_b a$, $T(n)$ is $\Theta(n^{\log_b a})$

$\Theta(n^{\log_b a})$, then $T(n)$ is $\Theta(n^{\log_b a} \lg n)$

$\Omega(n^c)$ for $c > \log_b a$, $T(n)$ is $\Theta(f(n))$

Max-Heapify

Runtime?

Obviously (is it?): lg n

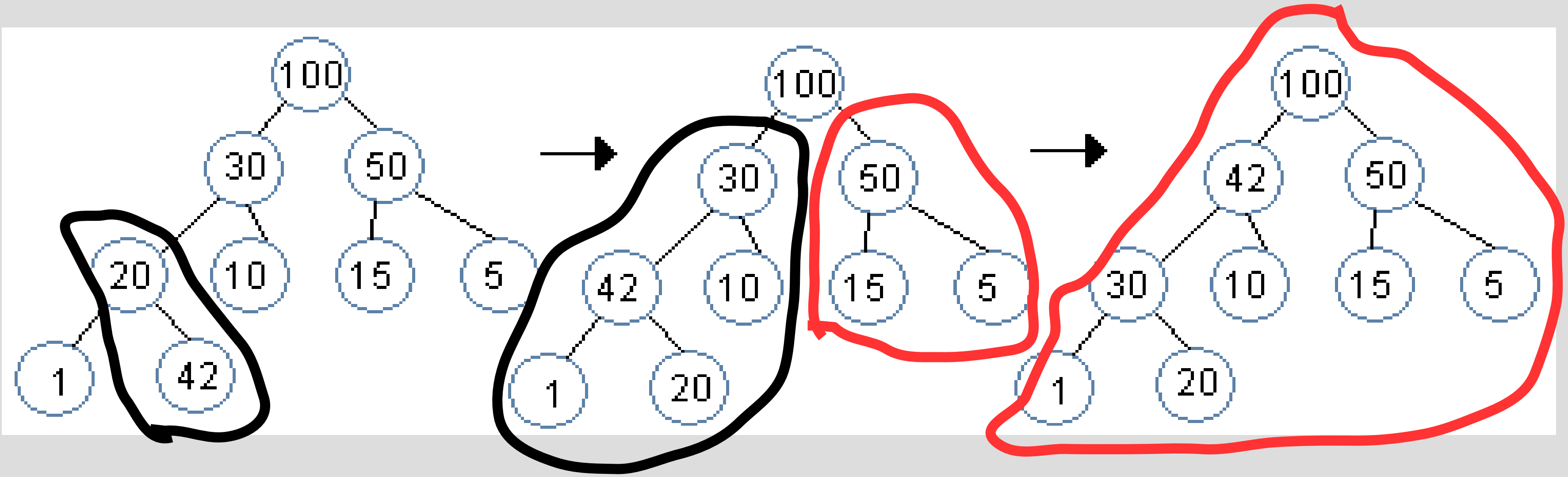
T(n) = T(2/3 n) + O(1) // why?
  Or...
T(n) = T(1/2 n) + O(1) = O(lg n)

Build-Max-Heap

Next we build a full heap from an unsorted sequence

```
Build-Max-Heap(A)
for i = floor( A.length/2 ) to 1
  Heapify(A, i)
```

Build-Max-Heap



Red part is already Heapified

Build-Max-Heap

Correctness:

Base: Each alone leaf is a max-heap

Step: if A[i] to A[n] are in a heap, then Heapify(A, i-1) will make i-1 a heap as well

Termination: loop ends at i=1, which is the root (so all heap)

Build-Max-Heap

Runtime?

Build-Max-Heap

Runtime?

O(n lg n) is obvious, but we can get a better bound...

Show ceiling($n/2^{h+1}$) nodes at any level 'h', with h=0 as bottom

Build-Max-Heap

Heapify from height 'h' takes O(h)

$\mathrm{sum}_{h=0}^{\lg n}\ \mathrm{ceiling}(n/2^{h+1})\ O(h)$

$=O(n\ \mathrm{sum}_{h=0}^{\lg n}\ \mathrm{ceiling}(h/2^{h+1}))$

$(\mathrm{sum}_{x=0}^{\infty}\ k\ x^k = x/(1-x)^2,\ x=1/2)$

$=O(n\ 4/2) = O(n)$

Heapsort

```
Heapsort(A):
Build-Max-Heap(A)
for i = A.length to 2
    Swap A[ 1 ], A[ i ]
    A.heapsize = A.heapsize – 1
    Max-Heapify(A, 1)
```

Heapsort

You try it!

Sort: A = [1, 6, 8, 4, 7, 3, 4]

Heapsort

First, build the heap starting here

A = [1, 6, 8, 4, 7, 3, 4]

A = [1, **6**, 8, 4, **7**, 3, 4]

A = [**1**, 7, **8**, 4, 6, 3, 4]

A = [8, 7, **1**, 4, 6, 3, **4**] - recursive

A = [8, 7, 4, 4, 6, 3, 1] - done

Heapsort

Move first to end, then re-heapify

A = [**8**, 7, 4, 4, 6, 3, **1**], move end

A = [**1**, **7**, 4, 4, 6, 3, 8], heapify

A = [7, **1**, 4, 4, **6**, 3, 8], rec. heap

A = [**7**, 6, 4, 4, 1, **3**, 8], move end

A = [**3**, **6**, 4, 4, 1, 7, 8], heapify

A = [6, **3**, 4, **4**, 1, 7, 8], rec. heap

A = [6, 4, 4, 3, 1, 7, 8], next slide..

Heapsort

A = [**6**, 4, 4, 3, **1**, 7, 8], move end

A = [**1**, 4, **4**, 3, 6, 7, 8], heapify

A = [**4**, 4, 1, **3**, 6, 7, 8], move end

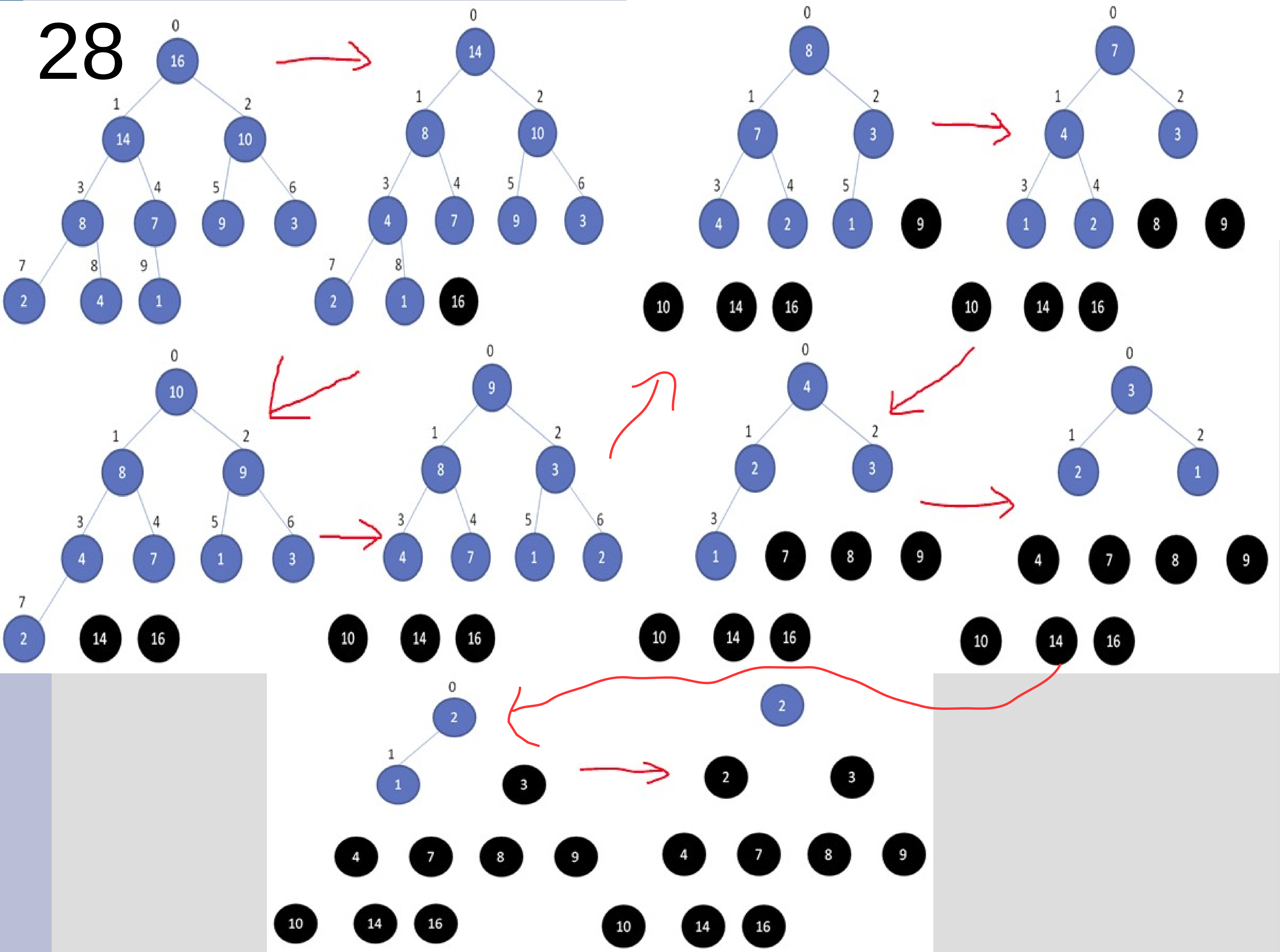
A = [**3**, **4**, 1, 4, 6, 7, 8], heapify

A = [**4**, 3, **1**, 4, 6, 7, 8], move end

A = [**1**, **3**, 4, 4, 6, 7, 8], heapify

A = [**3**, **1**, 4, 4, 6, 7, 8], move end

A = [1, 3, 4, 4, 6, 7, 8], done



Heapsort

Runtime?

Heapsort

Runtime?

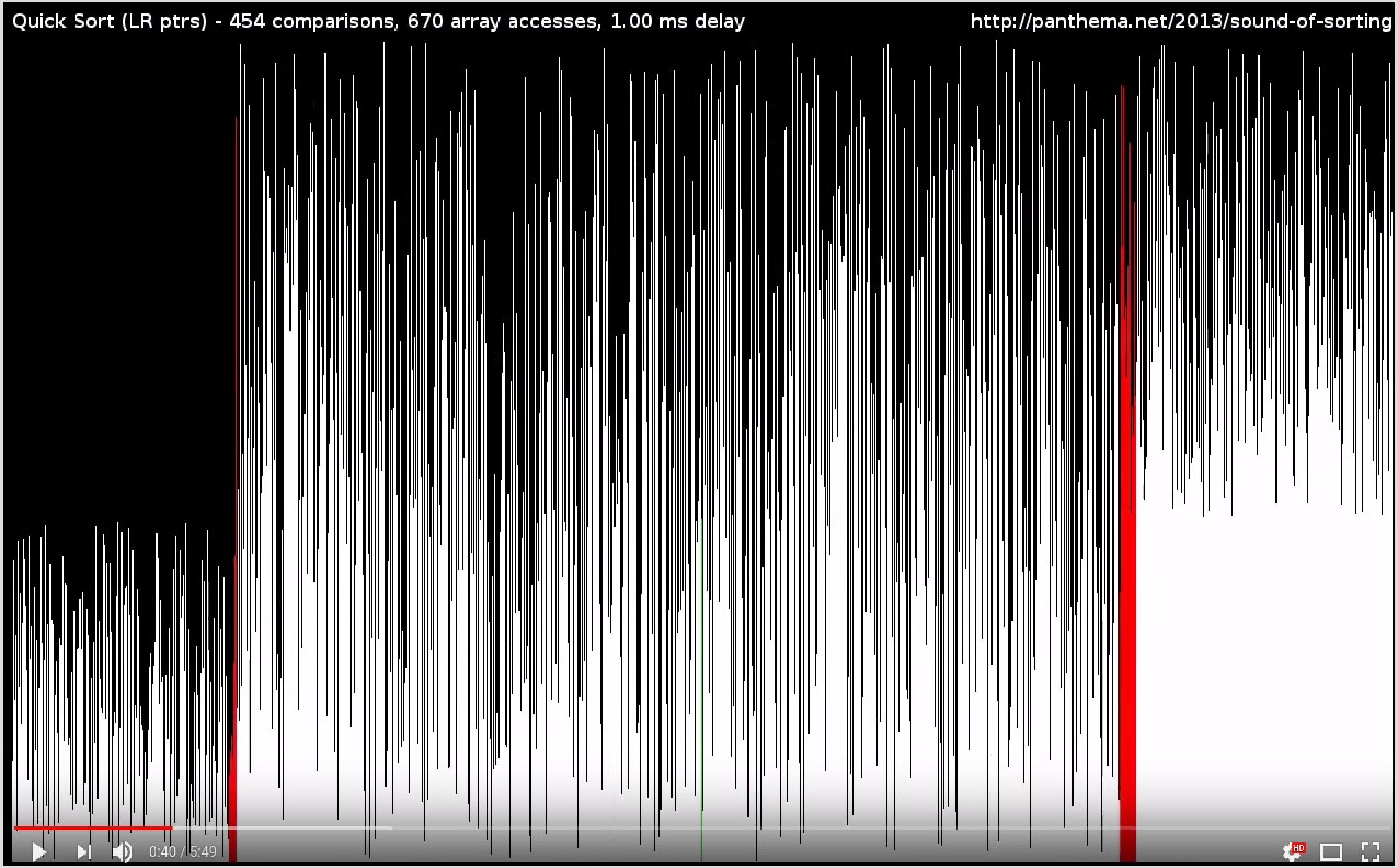
Run Max-Heapify O(n) times
So... O(n lg n)

Sorting comparisons:

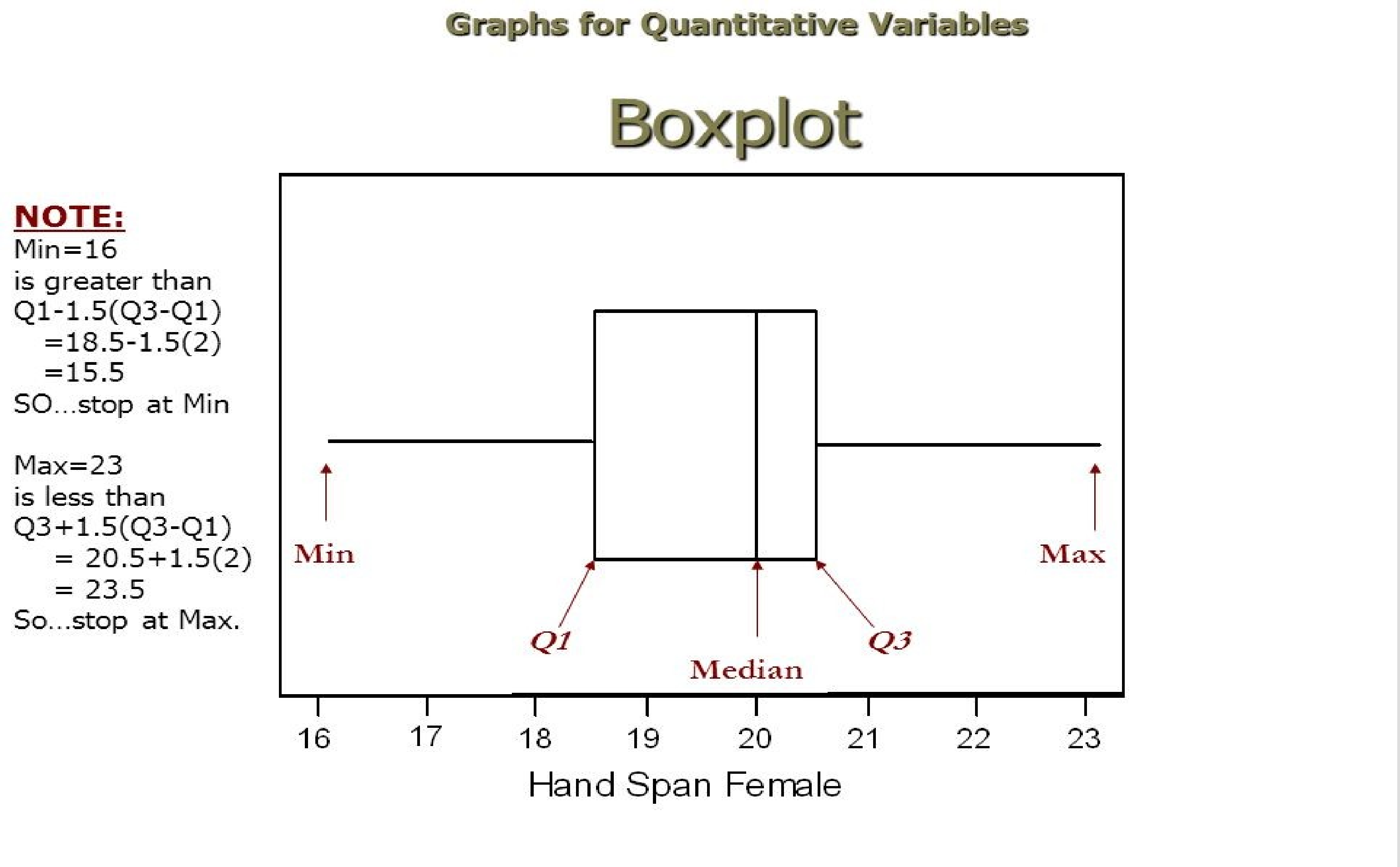
Name	Average	Worst-case
Insertion[s,i]	$O(n^2)$	$O(n^2)$
Merge[s,p]	$O(n \lg n)$	$O(n \lg n)$
Heap[i]	$O(n \lg n)$	$O(n \lg n)$
Quick[p]	$O(n \lg n)$	$O(n^2)$
Counting[s]	$O(n + k)$	$O(n + k)$
Radix[s]	$O(d(n+k))$	$O(d(n+k))$
Bucket[s,p]	$O(n)$	$O(n^2)$

Sorting comparisons:

https://www.youtube.com/watch?v=kPRA0W1kECg



Selection


Graphs for Quantitative Variables
Boxplot
NOTE:
Min=16
is greater than
Q1-1.5(Q3-Q1)
=18.5-1.5(2)
=15.5
SO...stop at Min
Max=23
is less than
Q3+1.5(Q3-Q1)
= 20.5+1.5(2)
= 23.5
So...stop at Max.
Min
Max
Q1
Median
Q3
16
17
18
19
20
21
22
23
Hand Span Female

Priority queues

Heaps can also be used to implement priority queues (i.e. airplane boarding lines)

Operations supported are: Insert, Maximum, Exctract-Max and Increase-key

Priority queues

```
Maximum(A):
  return A[ 1 ]


Extract-Max(A):
max = A[1]
A[1] = A.heapsize
A.heapsize = A.heapsize – 1
Max-Heapify(A, 1),    return max
```

Priority queues

```
Increase-key(A, i, key):
A[ i ] = key
while ( i>1 and A [floor(i/2)] < A[i])
   swap A[ i ], A [floor(i/2)]
   i = floor(i/2)
```

Opposite of Max-Heapify... move high keys up instead of low down

Priority queues

Insert(A, key):

A.heapsize = A.heapsize + 1

A [A.heapsize] = -∞

Increase-key(A, A.heapsize, key)

Priority queues

Runtime?

Maximum =
Extract-Max =
Increase-Key =
Insert =

Priority queues

Runtime?

Maximum = $O(1)$
Extract-Max = $O(\lg n)$
Increase-Key = $O(\lg n)$
Insert = $O(\lg n)$

Selection

Selection given a set of (distinct) elements, finding the element larger than $i - 1$ other elements

Selection with...
$i=n$ is finding maximum
$i=1$ is finding minimum
$i=n/2$ is finding median

Maximum

Selection for any i is O(n) runtime

Find max in O(n)?

Maximum

Selection for any i is O(n) runtime

Find max in O(n)?

```
max = A[ 1 ]
for i = 2 to A.length
   if ( A[ i ] > max )
      max = A[ i ]
```

Max and min

It takes about n comparisons to find max

How many would it take to find both max and min at same time?

Max and min

It takes about n comparisons to find max

How many would it take to find both max and min at same time?

Naïve = 2n
Smarter = 3/2 n

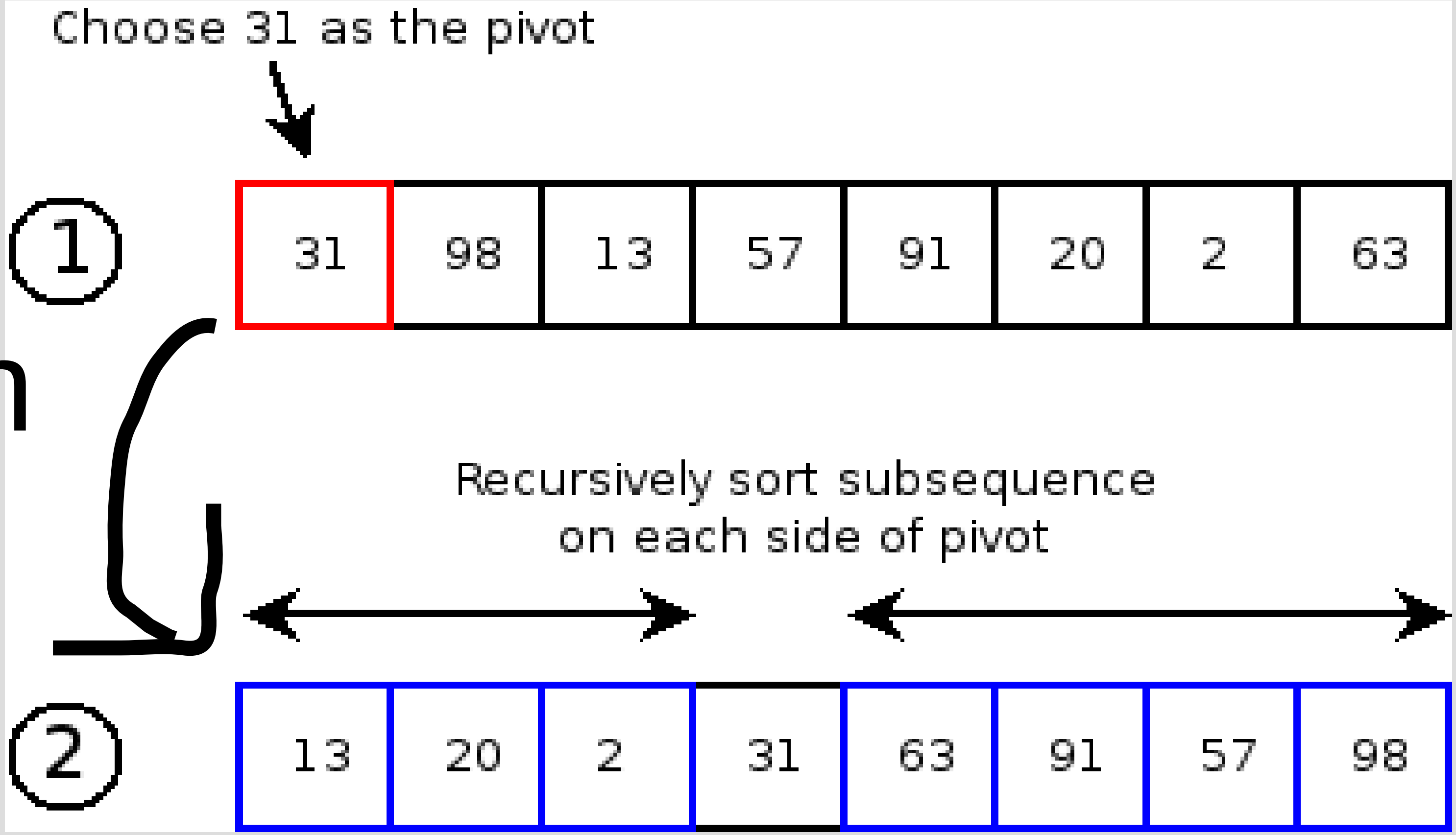
Max and min

```
smin = min(A[ 1 ], A[ 2 ])
smax = max(A[ 1 ], A[ 2 ])
for i = 3 to A.length step 2
   if (A[ i ] > A[ i+1 ])
      smax = max(A[ i ], smax)
      smin = min(A[ i+1], smin)
   else
      smax = max(A[ i+1], smax)
      smin = min(A[ i ], smin)
```

Randomized selection

Remember quicksort?

Partition step



Randomized selection

To select i:

1. Partition on random element

2. If partitioned element i, end otherwise recursively partition on side with i

Randomized selection

{2, 6, 4, 7, 8, 4, 7, 2} find i = 5

Pick pivot = 4

{2, 6, 4, 7, 8, 2, 7, 4}

{2, 6, 4, 7, 8, 2, 7, 4}

{2, 6, 4, 7, 8, 2, 7, 4}

{2, 4, 6, 7, 8, 2, 7, 4}

{2, 4, 6, 7, 8, 2, 7, 4}

{2, 4, 6, 7, 8, 2, 7, 4}

Randomized selection

{2, 4, 6, 7, 8, 2, 7, 4}

{2, 4, 2, 7, 8, 6, 7, 4}

{2, 4, 2, 7, 8, 6, 7, 4}

{2, 4, 2, 4, 7, 8, 6, 7}

1, 2, 3, 4, 5, 6, 7, 8

i=5 on green side, recurse

Randomized selection

{7, 8, 6, 7} pick pivot = 6

{7, 8, 7, 6}

{7, 8, 7, 6}

{7, 8, 7, 6}

{7, 8, 7, 6}

{6, 7, 8, 7}

5, 6, 7, 8

found i=5, value = 6

Randomized selection

Quicksort runs in $O(n \lg n)$, but we only have sort one side and sometimes stop early

This gives randomized selection $O(n)$ running time (proof in book, I punt)

Randomized selection

Just like quicksort, the worst case running time is $O(n^2)$

This happens when you want to find the min, but always partition on the max

Select

A worst case O(n) selection is given by Select: (see code)

1. Make n/5 groups of 5 and find their medians (via sorting)
2. Recursively find the median of the n/5 medians (using Select)
3. Partition on median of medians
4. Recursively Select correct side

Select

Proof of the general case:

$T(n) = \text{sum}_i\ T(k_i n + q_i) + O(n)$

// assume T(n) is O(n)

$T(n) = cn – cn+c\ \text{sum}_i(k_i n + q_i)+an$

so T(n) is O(n) if:

$– cn+c\ \text{sum}_i(k_i n + q_i)+an \leq 0$

$an \leq c(\ n\ (1 - \text{sum}_i\ k_i) - \text{sum}_i\ q_i)$

Select

$an \leq c(\ n\ (1 - \mathrm{sum}_i\ k_i) - \mathrm{sum}_i\ q_i)$

$an/(n(1-\mathrm{sum}_i\ k_i) - \mathrm{sum}_i\ q_i) \leq c$

// Pick $n > 2(\mathrm{sum}_i\ q_i/(1 - \mathrm{sum}_i\ k_i))$

$c \geq a\ 2(\mathrm{sum}_i\ q_i/(1-\mathrm{sum}_i\ k_i))/(\mathrm{sum}_i\ q_i)$

$c \geq 2\ a\ /\ (1 - \mathrm{sum}_i\ k_i)$

Done as $\mathrm{sum}_i\ k_i < 1$

Select

Select runs in:

$T(n) = T(\text{ceiling}(n/5))$
$\quad + T(7n/10 + 6) + O(n)$

By the previous proof this is O(n):

$\text{ceiling}(n/5) + 7n/10 + 6$
$\leq n/5 + 1 + 7n/10 + 6 = 9n/10 + 7$
$\text{sum}_i\ k_i = 9/10 < 1$, done

Select

Does this work for making:

(1) n/3 groups of 3?

(2) n/7 groups of 7?

(3) n/9 groups of 9?