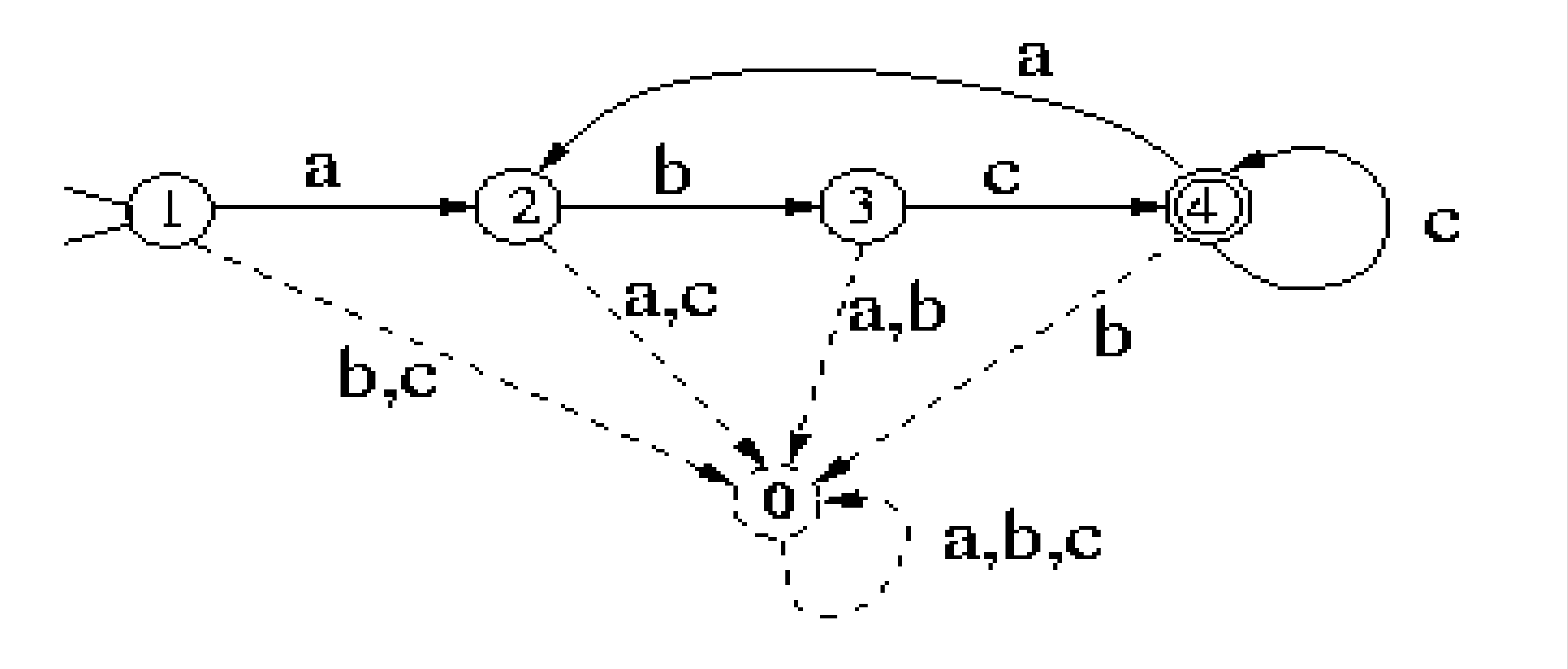
# String matching



# Announcements

Programming assignment
extended to Thursday

# Prefix vs suffix

w is a prefix of x=w [ x, means exists y s.t. wy = x (also implies $|w| \leq |x|$)
(w ] y = w is a suffix of x)

An easy way to remember prefix vs suffix is: prefix = [, which looks like beginning of an array (similar suffix)

# Finite Automata

A finite automata has 5 parts:

(1) A set of states Q

(2) An initial state $q_0$

(3) Some accepting states, A subset Q

(4) An alphabet, $\sum$

(5) A transition function $\delta$, from

$Q \times \sum$ to Q, namely $\delta(q,a)=\sigma(P_q a)$

# Finite Automata

Let $\sigma(x) = \max \{k : P_k\ ]\ x\}$

So $\sigma$ is the longest prefix of P that is also a suffix of x:

$P = \{a, b, a, a, b, c, a\}$

$\sigma(b\ a\ a\ c\ b\ a\ b) = 2$ (all longer bad)

# Finite Automata

Compute-Transition-Function(P, ∑)

for q = 0 to |P|

  for each a in ∑

    k = min( |P|, q+1 )//end P or q

    while: not $P_k$ ] $P_q$a

      k = k – 1

    δ(q,a) = k                    // runtime?

# Finite Automata

$O(|P|^3|\Sigma|)$, but can get to $O(|P|\,|\Sigma|)$ if smart

$|P|$ - outside loop

$|\Sigma|$ - outside loop

$|P|$ - repeat runs at most $|P|$ times

$|P|$ - $P_k$ ] $P_q$a checks $O(|P|)$ chars

# Finite Automata

```
Finite-Automaton-Matcher(T,δ,|P|)
q=0 // q is state
for i = 1 to |T|
   q = δ(q,T[ i ])
   if q == |P|
      print "Pattern occurs at shift" i-|P|
```

Runtime = O(|T|)

# Finite Automata

1, 2, 3, 4, 5, 6, 7

P = {a, b, a, a, b, c, a}, then δ is:

| | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|---|
| a | 1 | 1 | 3 | 4 | 1 | 3 | 7 | 1 |
| b | 0 | 2 | 0 | 2 | 5 | 0 | 0 | 2 |
| c | 0 | 0 | 0 | 0 | 0 | 6 | 0 | 0 |

(see FAsigma.py)

# Finite Automata

| | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|---|
| a | 1 | 1 | 3 | 4 | 1 | 3 | 7 | 1 |
| b | 0 | 2 | 0 | 2 | 5 | 0 | 0 | 2 |
| c | 0 | 0 | 0 | 0 | 0 | 6 | 0 | 0 |

S = {a, b, c, a, b, a, a, b, c, a, c, a}

Start 0, see $S_1$='a', goto 1...

At 1, see $S_2$='b', goto 2...

At 2, see $S_3$='c', goto 0...

At 0, see $S_4$='a', goto 1...

# Finite Automata

| | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|---|
| a | 1 | 1 | 3 | 4 | 1 | 3 | 7 | 1 |
| b | 0 | 2 | 0 | 2 | 5 | 0 | 0 | 2 |
| c | 0 | 0 | 0 | 0 | 0 | 6 | 0 | 0 |

S = {a, b, c, a, b, a, a, b, c, a, c, a}

At 1, see $S_5$='b', goto 2...

At 2, see $S_6$='a', goto 3...

At 3, see $S_7$='a', goto 4...

At 4, see $S_8$='b', goto 5...

# Finite Automata

| | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|---|
| a | 1 | 1 | 3 | 4 | 1 | 3 | 7 | 1 |
| b | 0 | 2 | 0 | 2 | 5 | 0 | 0 | 2 |
| c | 0 | 0 | 0 | 0 | 0 | 6 | 0 | 0 |

S = {a, b, c, a, b, a, a, b, c, a, c, a}

At 5, see $S_9$='c', goto 6...

At 6, see $S_{10}$='a', goto 7... MATCH!

At 7, see $S_{11}$='a', goto 1...

At 1, see $S_{12}$='c', goto 0...

# Finite Automata

| | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|---|
| a | 1 | 1 | 3 | 4 | 1 | 3 | 7 | 1 |
| b | 0 | 2 | 0 | 2 | 5 | 0 | 0 | 2 |
| c | 0 | 0 | 0 | 0 | 0 | 6 | 0 | 0 |

S = {a, b, c, a, b, a, a, b, c, a, c, a}

At 0, see $S_{13}$='a', goto a...

Done, one match found ending at $S_{10}$ (so match starts $S_4$)

# Finite Automata

You try it!

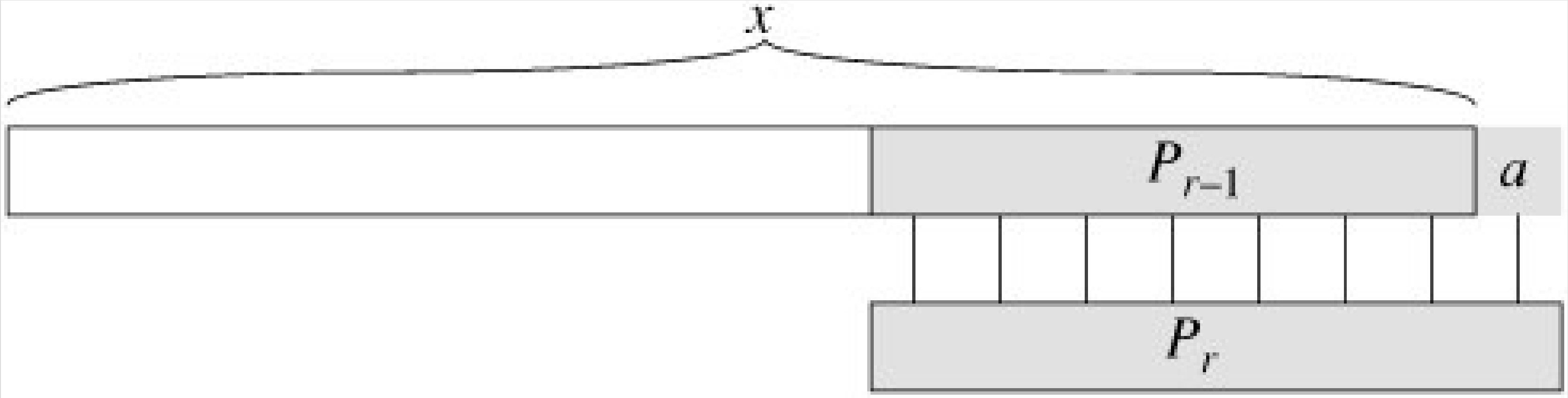
P={a, b, a, a}
S={a, a, b, a, c, a, a, b, a, a, b, a, a, a}

What is automata?
Where are matches?

# FA correctness

Lemma 32.2: $\sigma(xa) \leq \sigma(x) + 1$

Obvious...



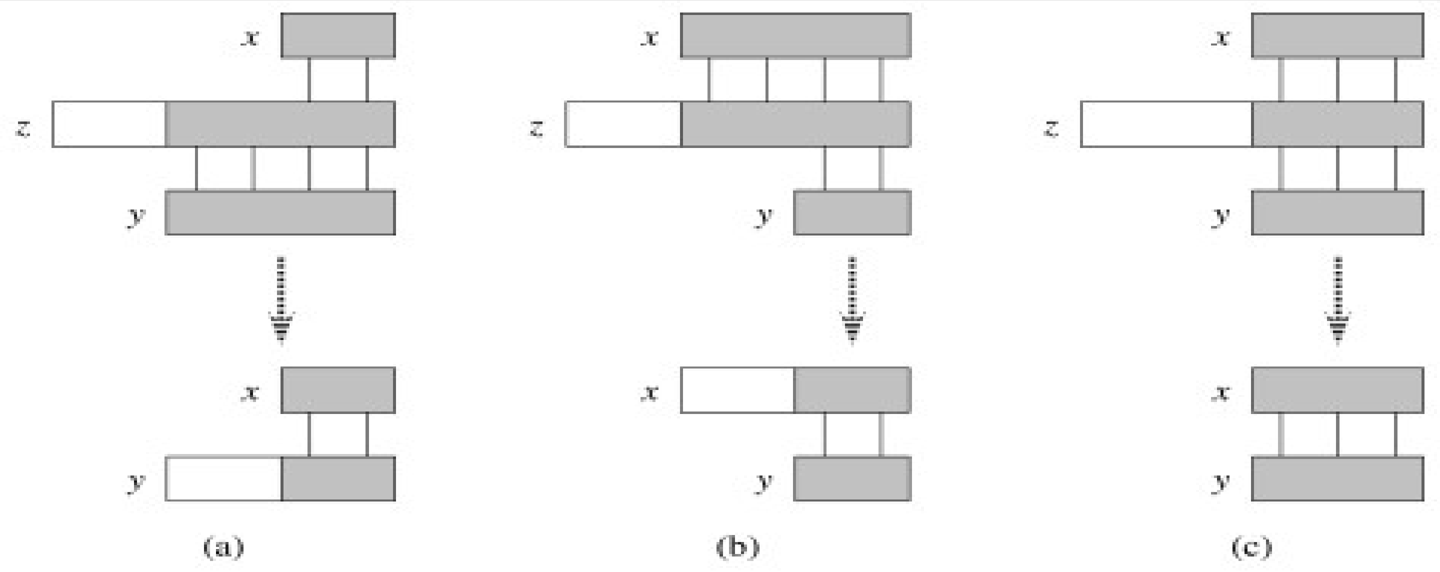
# FA correctness

If x ] z and y ] z, then:

(a) If $|x| \leq |y|$, x ] y

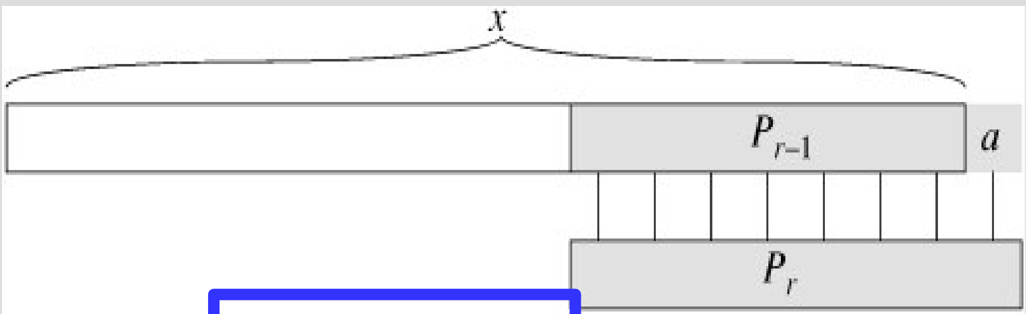
(b) If $|y| \leq |x|$, y ] x

(c) If $|x| = |y|$, x = y



# FA correctness

Lemma 32.3: if $q=\sigma(x)$, then $\sigma(xa)=\sigma(P_q a)$




Proof:

$P_q$ ] x by def of $q=\sigma(x)$, then $\boxed{P_q a \text{ ] } xa}$

Let $r=\sigma(xa)$ then $P_r$ ] xa and $r \le q+1$

So $|P_r| \le |P_q a|$ means $P_r$ ] $P_q a$

$\sigma(xa)\le\sigma(P_q a)$,

$\boxed{P_q a \text{ ] } xa,}$ so also $\sigma(P_q a)\le\sigma(xa)$, thus equal

# FA correctness

Theorem 32.4: if $\Phi$ is the final-state function, then $\Phi(T_i) = \sigma(T_i)$

Base: $T_0 = \varepsilon$, so $\Phi(T_0)=0=\sigma(T_0)$

Induction: $\Phi(T_{i+1}) = \Phi(T_i a) = \delta(\Phi(T_i),a) = \sigma(P_q a) = \sigma(T_i a) = \sigma(T_{i+1})$, where $q=\Phi(T_i)$

# Knuth-Morris-Pratt

Faster computation by using pattern symmetries within itself (vs transitions for each char/state)

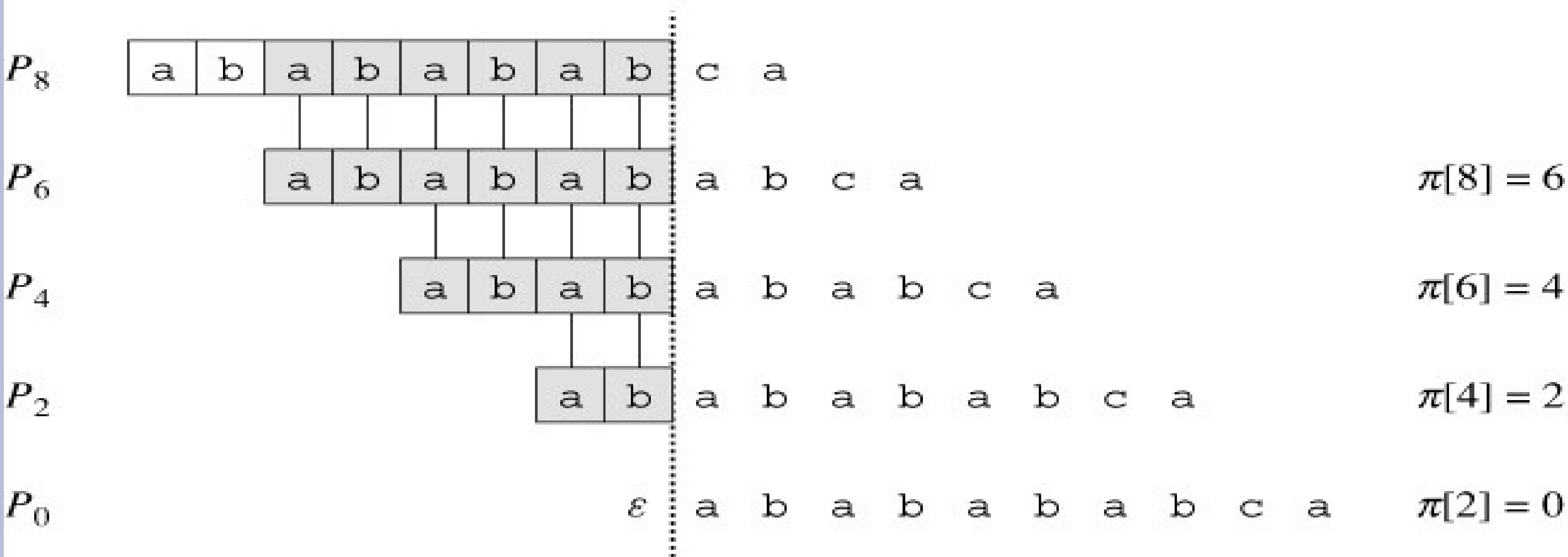
The function $\pi$ does this, namely
$\pi(q) = \max(k : k < q \text{ and } P_k \,]\, P_q)$

Namely, $\pi$ finds shifts of P on itself

# Knuth-Morris-Pratt

| $i$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| $P[i]$ | a | b | a | b | a | b | a | b | c | a |
| $\pi[i]$ | 0 | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 0 | 1 |

(a)




(b)

# Knuth-Morris-Pratt

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| i | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
| P[i] | a | b | a | a | b | c | a |
| $\pi(i)$ | 0 | 0 | 1 | 1 | 2 | 0 | 1 |

(See: FAsigma.py ... again)

# Knuth-Morris-Pratt

```
KMP-Matcher(T,P,π) // runtime?
q = 0
for i = 1 to |T|
  while q > 0 and P[q+1] ≠ T[ i ]
    q = π[q]
  if P[q+1] == T[ i ], then q = q+1
  if q == |P|
    match found, and set q = π[q]
```

# Knuth-Morris-Pratt

The while loop decreases q, so it can only run as many times as q increases

q increases only if match in T, so at most |T| times

O(|T| + |T|) = O(|T|)
(why not |T|*|T|?)

# Knuth-Morris-Pratt

```
Compute-Prefix-Function(P)
k = 0, π[1] = 0
for q = 2 to |P|
  while k > 0 and P[k+1] ≠ P[q]
    k = π[k]
  if P[k+1] == P[q]
    k = k+1
  π[q]=k            // Runtime = O(|P|)
```

# KMP correctness

Let $\pi^*[q] = \{\pi[q], \pi[\pi[q]], \ldots 0\}$

Lemma 32.5: $\pi^*[q] = \{k : k < q$ and $P_k$ ] $P_q$ $\}$

Remember:

$\pi(q) = \max(k : k < q$ and $P_k$ ] $P_q)$,

so fairly obvious (see next slide)

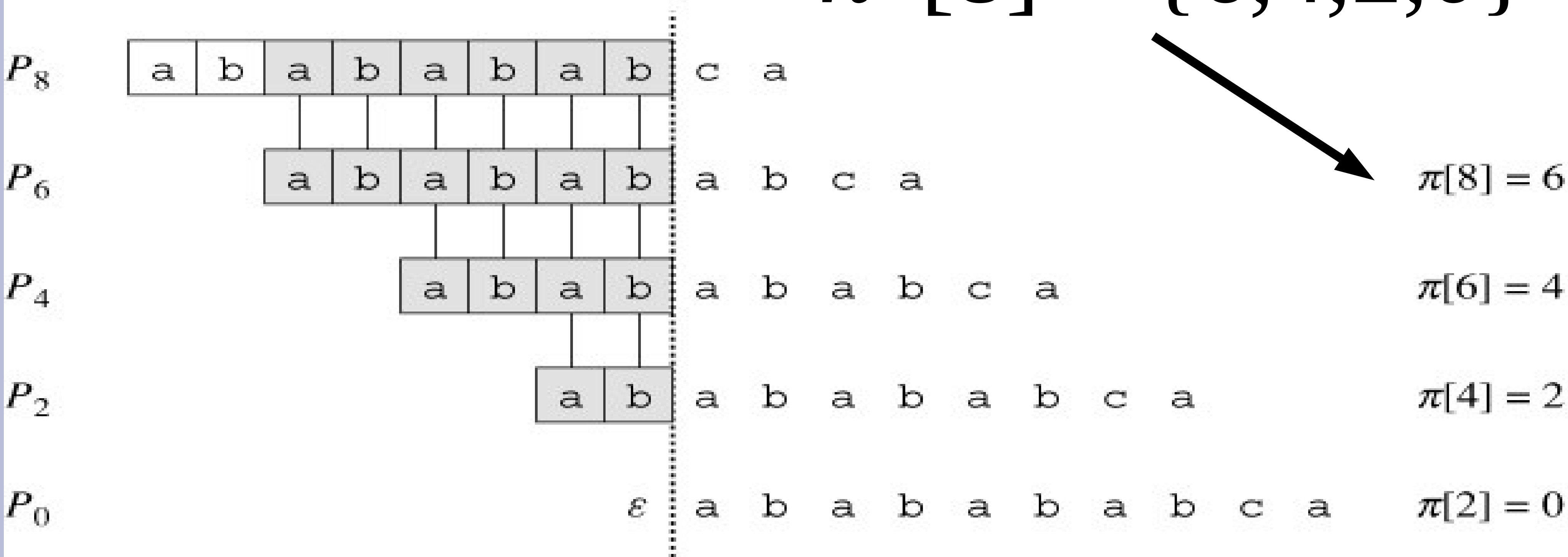
(Tip: prove 2 sets equal by showing A subset B and B subset A)

# KMP correctness

| $i$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| $P[i]$ | a | b | a | b | a | b | a | b | c | a |
| $\pi[i]$ | 0 | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 0 | 1 |

(a)

$\pi^*[8] = \{6,4,2,0\}$




(b)

# KMP correctness

Lemma 32.6: if $\pi[q] > 0$, then $\pi[q]-1$ in $\pi^*[q-1]$

Proof: $\pi[q] < q$ and $P_{\pi[q]}$ ] $P_q$, so $\pi[q] - 1 < q - 1$ and $P_{\pi[q]-1}$ ] $P_{q-1}$ (we know $\pi[q] > 0$, so we can drop a char)

Previous lemma says: $\pi^*[q] = \{k :$ $k < q$ and $P_k$ ] $P_q$ $\}$, above let $q=q-1$, $k=\pi[q]-1$, then done

# KMP correctness

Let $E_{q-1}$={k in $\pi$*[q-1] : P[k+1]=P[q]}

Corollary 32.7: $\pi$[q] = {0 or 1+max{k in $E_{q-1}$} if $E_{q-1}$ not empty}

Proof:

Case 1: $E_{q-1}$ empty, no match, so 0

Case 2: By def of $E_{q-1}$, k+1 < q and $P_{k+1}$]$P_q$ implies $\pi[q] \geq 1+\max\{k \text{ in } E_{q-1}\}$

# KMP correctness

($E_{q-1}$={k in $\pi^*[q-1]$ : P[k+1]=P[q]})

Case 2 (cont): $\pi[q] \geq 1+\max\{k \text{ in } E_{q-1}\}$

Let r = $\pi[q] - 1$, then $P_{r+1}$ ] $P_q$ so

P[r+1] = P[q]. Lemma 32.6 says

r in $\pi^*[q-1]$, so r in $E_{q-1}$.

Thus $\pi[q] \leq 1+\max\{k \text{ in } E_{q-1}\}$

Thus $\pi[q] = 1+\max\{k \text{ in } E_{q-1}\}$

# KMP correctness

$k=\pi[q-1]$ at the start of the for loop in Compute-Prefix-Function alg

The while loop finds max{k in $E_{q-1}$} and adds one for Corollary 32.7

If there $k=0$, then either the max was 0 and it will be incremented to 1 or no match and will stay 0

# KMP correctness

KMP alg correctness (map to FA alg):

Base: both start with q=0

Step ($q'=\sigma(T_{i-1})$):

Case $\sigma(T_i)=0$: q=0 and same

Case $\sigma(T_i)=q'+1$: while does not run, then increases q, so $q=q'+1=\sigma(T_i)$

(continued)

# KMP correctness

Step: $q'=\sigma(T_{i-1})$, Case $0<\sigma(T_i)<q'$:

while loop terminates when

$P[q+1]=T[i]$, so $q+1 = \sigma(P_{q'}T[i])$

$=\sigma(T_{i-1}T[i])$

$=\sigma(T_i)$, then q is incremented so...

$q=\sigma(T_i)$