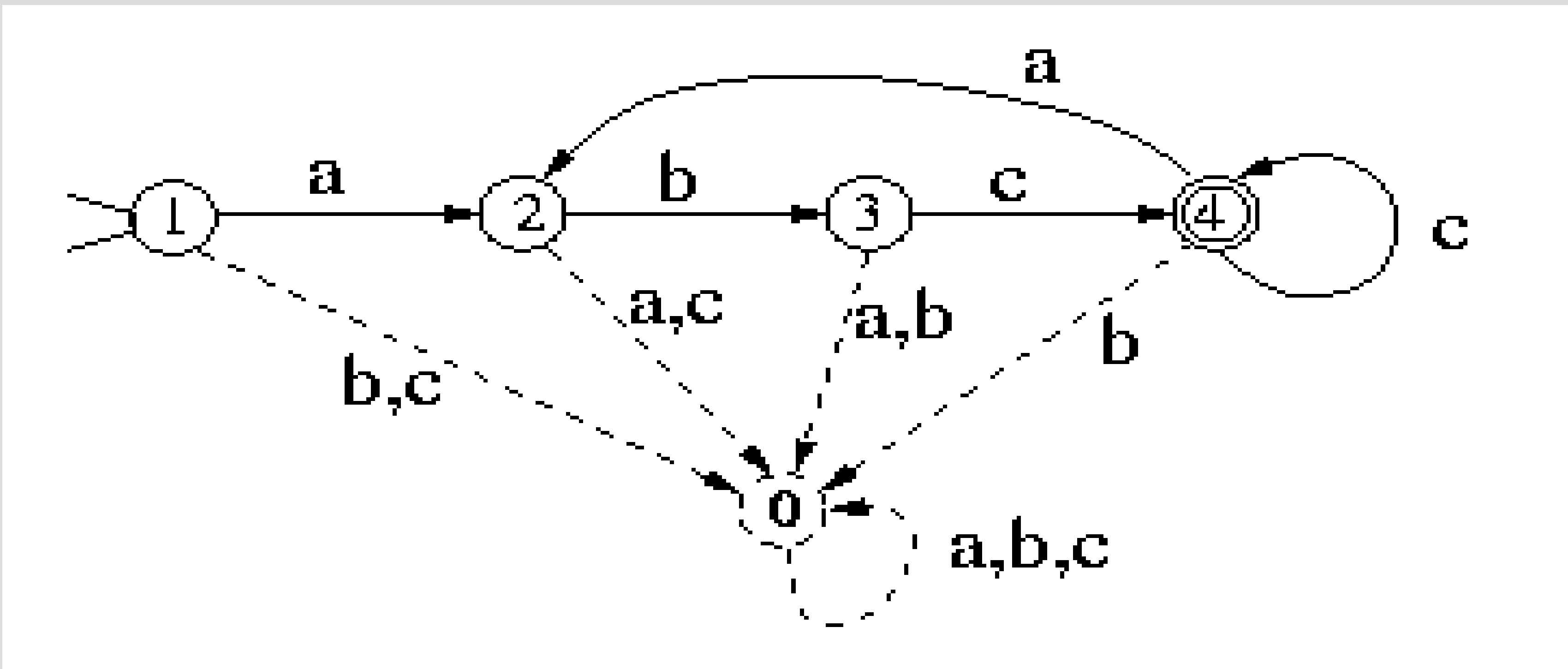
String matching



Announcements

Programming assignment
extended to Thursday

Exam next week:
open book/notes
Covers: sorting, selection, greedy algorithms

Prefix vs suffix

$\sqsubset$

w is a prefix of x=w [x, means exists y s.t. wy = x (also implies |w| ≤ |x|)
(w] y = w is a suffix of x)

An easy way to remember prefix vs suffix is: prefix = [, which looks like beginning of an array (similar suffix)

Finite automata

We built a graph, where arro s are:

$\delta(q,a)=\sigma(P_q a)$

... with:

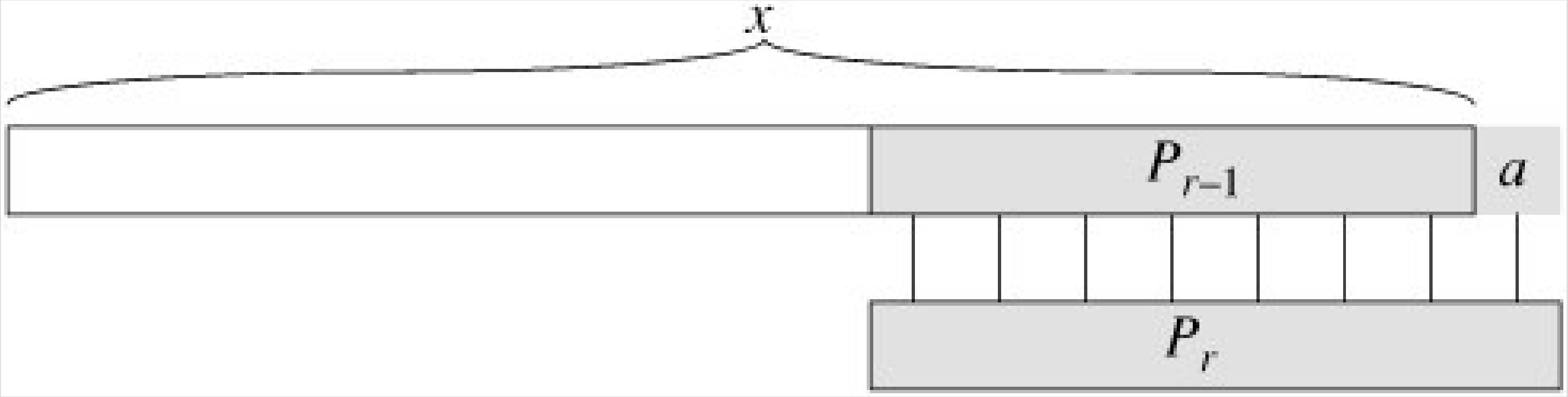
$\sigma(x) = \max\{k : P_k] x\}$

Today we will prove corectness!

FA correctness

Lemma 32.2: $\sigma(xa) \leq \sigma(x) + 1$

Obvious...



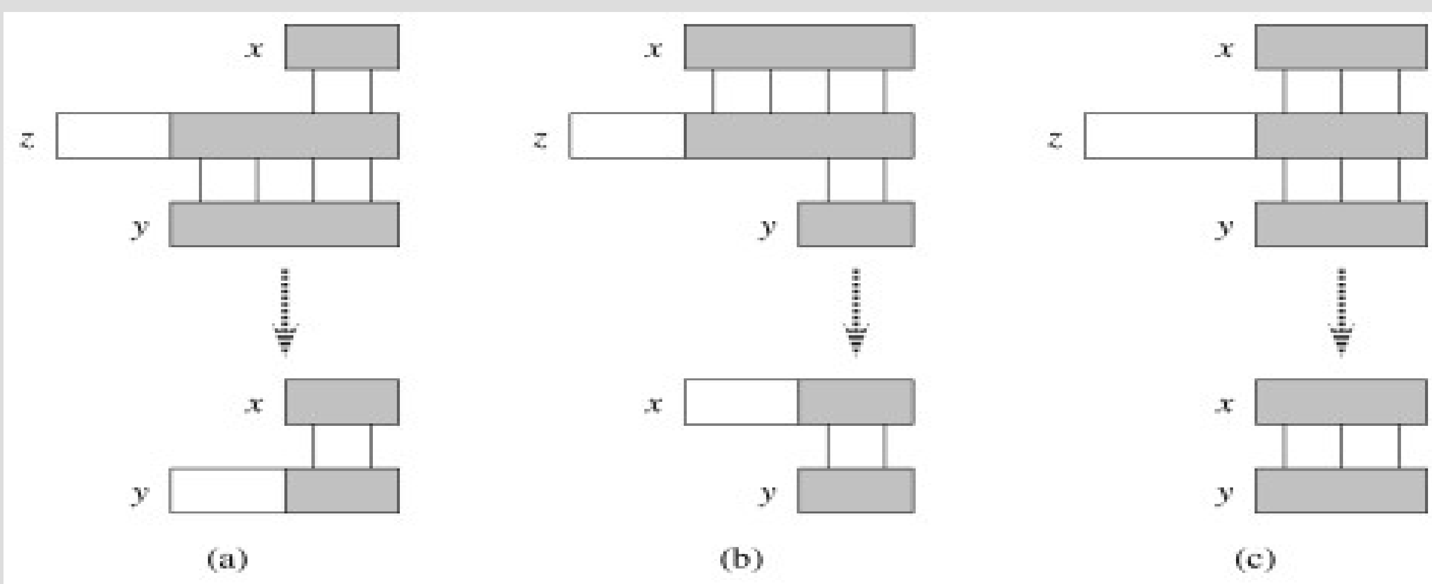
FA correctness

If x] z and y] z, then:

(a) If $|x| \leq |y|$, x] y

(b) If $|y| \leq |x|$, y] x

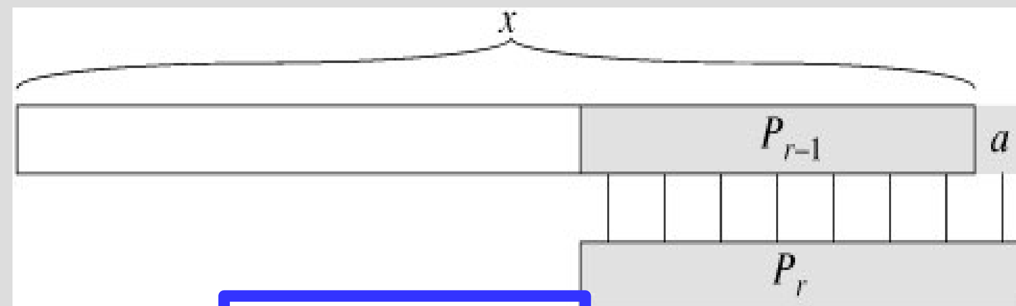
(c) If $|x| = |y|$, x = y



FA correctness

Lemma 32.3: if $q=\sigma(x)$,

then $\sigma(xa)=\sigma(P_q a)$



Proof:

P_q] x by def of $q=\sigma(x)$, then $\boxed{P_q a \,]\, xa}$

Let $r=\sigma(xa)$ then P_r] xa and $r \le q+1$

So $|P_r| \le |P_q a|$ means P_r] $P_q a$

$\sigma(xa)\le\sigma(P_q a)$,

$\boxed{P_q a \,]\, xa,}$ so also $\sigma(P_q a)\le\sigma(xa)$, thus equal

FA correctness

Theorem 32.4: if Φ is the final-state function, then $\Phi(T_i) = \sigma(T_i)$

Base: $T_0 = \varepsilon$, so $\Phi(T_0)=0=\sigma(T_0)$

Induction: $\Phi(T_{i+1}) = \Phi(T_i a) = \delta(\Phi(T_i),a) = \sigma(P_q a) = \sigma(T_i a) = \sigma(T_{i+1})$, where $q=\Phi(T_i)$

Knuth-Morris-Pratt

Faster computation by using pattern symmetries within itself (vs transitions for each char/state)

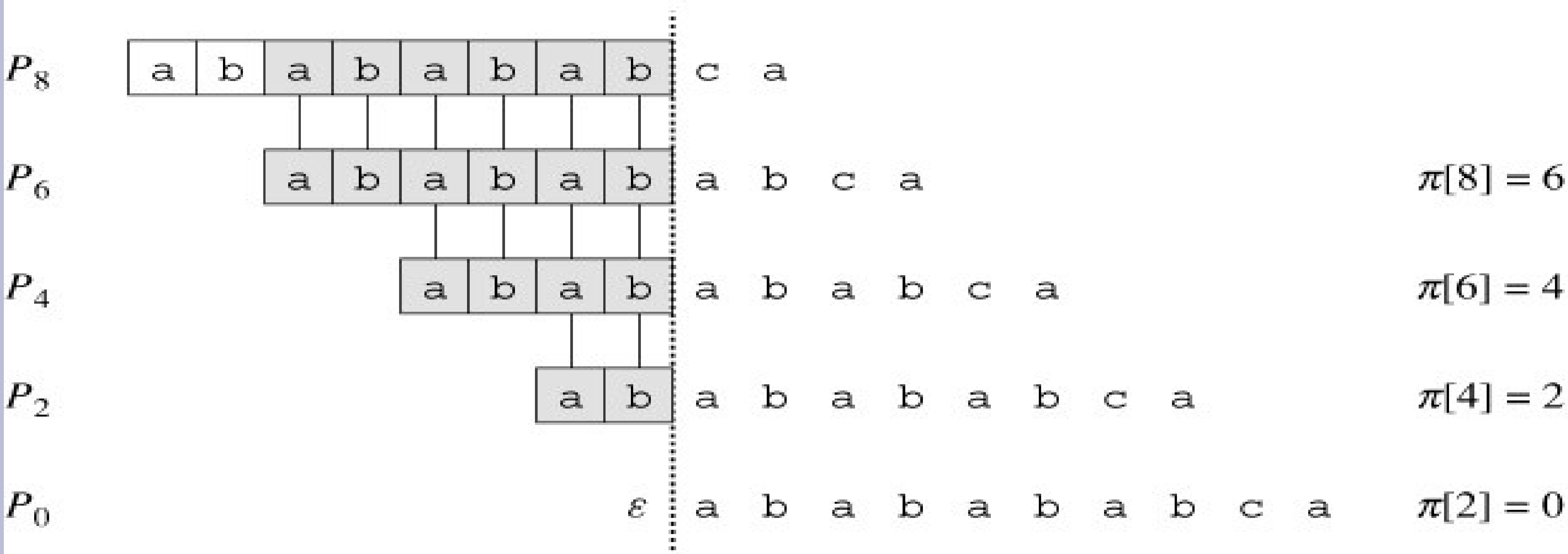
The function π does this, namely
$\pi(q) = \max(k : k < q \text{ and } P_k\ ]\ P_q)$

Namely, π finds shifts of P on itself

Knuth-Morris-Pratt

i	1	2	3	4	5	6	7	8	9	10
$P[i]$	a	b	a	b	a	b	a	b	c	a
$\pi[i]$	0	0	1	2	3	4	5	6	0	1

(a)



(b)

Knuth-Morris-Pratt

Let's look at the example from last time:

P = “abaabca”
T = “abcabaabcaca”

First we need to compute π's:
Find shifts of pattern with itself

Knuth-Morris-Pratt

i	1	2	3	4	5	6	7
P[i]	a	b	a	a	b	c	a
$\pi(i)$	0	0	1	1	2	0	1

Now we just need to run through the string T...

(See: FAsigma.py ... again)

Knuth-Morris-Pratt

T = “abcabaabcaca”

i	1	2	3	4	5	6	7
P[i]	a	b	a	a	b	c	a
π(i)	0	0	1	1	2	0	1

Start q=0, see T[1]='a'=P[q+1]=P[1]

At q=1, see T[2]='b'=P[q+1]=P[2]

At q=2, see T[3]='c'... not P[q+1]

π(q) = π(2) = 0. At 0, stop follow π

At q=0, see T[4]='a'=P[q+1]=P[1]

At q=1, see T[5]='b'=P[q+1]=P[2]

Knuth-Morris-Pratt

T = “abcabaabcaca”

i	1	2	3	4	5	6	7
P[i]	a	b	a	a	b	c	a
π(i)	0	0	1	1	2	0	1

At q=1, see T[5]='b'=P[q+1]=P[2]

At q=2, see T[6]='a'=P[q+1]=P[3]

At q=3, see T[7]='a'=P[q+1]=P[4]

At q=4, see T[8]='b'=P[q+1]=P[5]

At q=5, see T[9]='c'=P[q+1]=P[6]

At q=6, see T[10]='a'=P[q+1]=p[7]

Knuth-Morris-Pratt

T = “abcabaabcaca”

i	1	2	3	4	5	6	7
P[i]	a	b	a	a	b	c	a
π(i)	0	0	1	1	2	0	1

At q=6, see T[10]='a'=P[q+1]=p[7]
Match! Set q=π(q)=π(7)=1
At q=1, see T[11]='c'... not P[2]
π(q) = π(1) = 0. At 0, stop follow π
At q=0, see T[12]='a'=P[q+1]=P[1]
At q=1, but no more T, so done

Knuth-Morris-Pratt

```
Compute-Prefix-Function(P)
k = 0, π[1] = 0
for q = 2 to |P|
  while k > 0 and P[k+1] ≠ P[q]
    k = π[k]
  if P[k+1] == P[q]
    k = k+1
  π[q]=k              // Runtime = ???
```

Knuth-Morris-Pratt

```
Compute-Prefix-Function(P)
k = 0, π[1] = 0
for q = 2 to |P|
  while k > 0 and P[k+1] ≠ P[q]
    k = π[k]
  if P[k+1] == P[q]
    k = k+1
  π[q]=k            // Runtime = O(|P|)
```

Knuth-Morris-Pratt

```
KMP-Matcher(T,P,π) // runtime?
q = 0
for i = 1 to |T|
   while q > 0 and P[q+1] ≠ T[ i ]
      q = π[q]
   if P[q+1] == T[ i ], then q = q+1
   if q == |P|
      match found, and set q = π[q]
```

Knuth-Morris-Pratt

The while loop decreases q, so it can only run as many times as q increases

q increases only if match in T, so at most |T| times

O(|T| + |T|) = O(|T|)
(why not |T|*|T|?)

Knuth-Morris-Pratt

You try it!

P={a, b, a, a}

S={a, a, b, a, c, a, a, b, a, a, b, a, a, a}

What are π's?

Where are matches?

KMP correctness

Let $\pi^*[q] = \{\pi[q], \pi[\pi[q]], \ldots 0\}$

Lemma 32.5: $\pi^*[q] = \{k : k < q$ and $P_k] P_q \}$

Remember:

$\pi(q) = \max(k : k < q$ and $P_k] P_q)$,

so fairly obvious (see next slide)

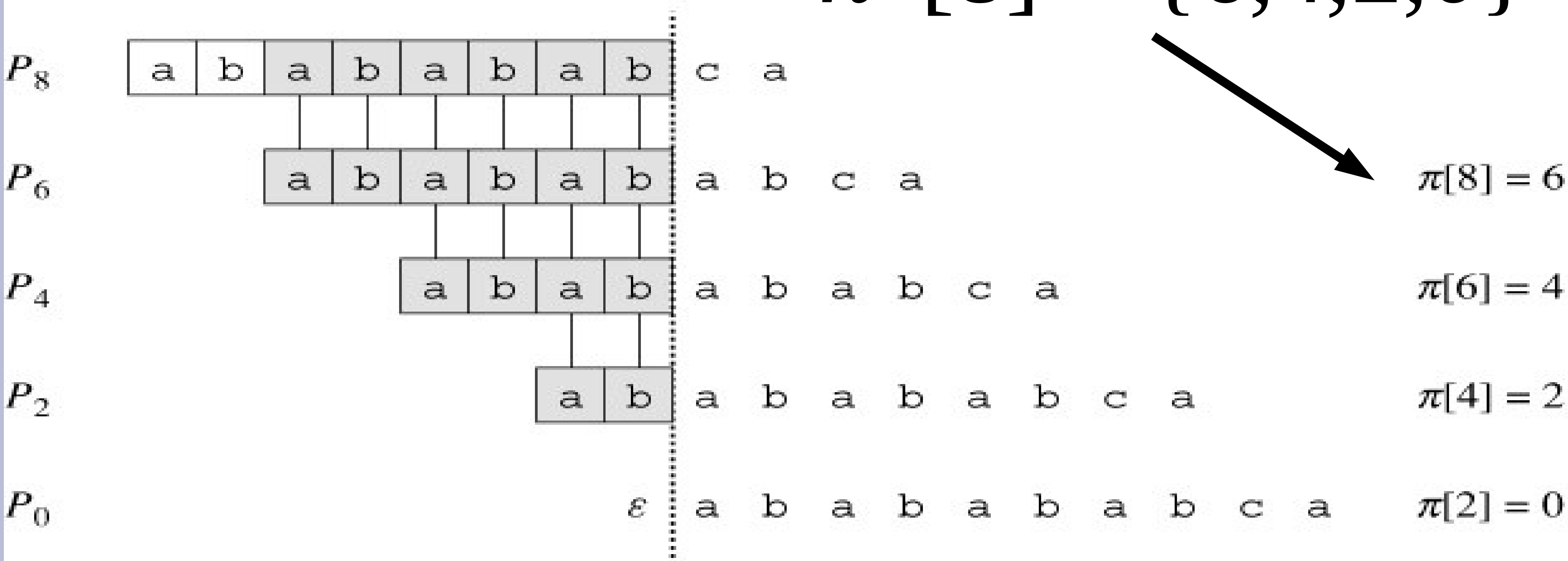
(Tip: prove 2 sets equal by showing A subset B and B subset A)

KMP correctness

i	1	2	3	4	5	6	7	8	9	10
$P[i]$	a	b	a	b	a	b	a	b	c	a
$\pi[i]$	0	0	1	2	3	4	5	6	0	1

(a)

$\pi^*[8] = \{6,4,2,0\}$



(b)

KMP correctness

Lemma 32.6: if $\pi[q] > 0$, then $\pi[q]-1$ in $\pi^*[q-1]$

Proof: $\pi[q] < q$ and $P_{\pi[q]}$] P_q, so $\pi[q] - 1 < q - 1$ and $P_{\pi[q]-1}$] P_{q-1} (we know $\pi[q] > 0$, so we can drop a char)

Previous lemma says: $\pi^*[q] = \{k : k < q$ and P_k] P_q $\}$, above let $q=q-1$, $k=\pi[q]-1$, then done

KMP correctness

Let E_{q-1}={k in π*[q-1] : P[k+1]=P[q]}

Corollary 32.7: π[q] = {0 or 1+max{k in E_{q-1}} if E_{q-1} not empty}

Proof:

Case 1: E_{q-1} empty, no match, so 0

Case 2: By def of E_{q-1}, k+1 < q and P_{k+1}]P_q implies $\pi[q] \geq 1+\max\{k \text{ in } E_{q-1}\}$

KMP correctness

$(E_{q-1} = \{k \text{ in } \pi^*[q-1] : P[k+1]=P[q]\})$

Case 2 (cont): $\pi[q] \geq 1+\max\{k \text{ in } E_{q-1}\}$

Let $r = \pi[q] - 1$, then P_{r+1}] P_q so

$P[r+1] = P[q]$. Lemma 32.6 says

r in $\pi^*[q-1]$, so r in E_{q-1}.

Thus $\pi[q] \leq 1+\max\{k \text{ in } E_{q-1}\}$

Thus $\pi[q]=1+\max\{k \text{ in } E_{q-1}\}$

KMP correctness

k=π[q-1] at the start of the for loop in Compute-Prefix-Function alg
The while loop finds max{k in E_{q-1}} and adds one for Corollary 32.7

If there k=0, then either the max was 0 and it will be incremented to 1 or no match and will stay 0

KMP correctness

KMP alg correctness (map to FA alg):

Base: both start with q=0

Step ($q'=\sigma(T_{i-1})$):

Case $\sigma(T_i)=0$: q=0 and same

Case $\sigma(T_i)=q'+1$: while does not run, then increases q, so $q=q'+1=\sigma(T_i)$

(continued)

KMP correctness

Step: $q'=\sigma(T_{i-1})$, Case $0<\sigma(T_i)<q'$:

while loop terminates when

$P[q+1]=T[i]$, so $q+1 = \sigma(P_{q'}T[i])$

$=\sigma(T_{i-1}T[i])$

$=\sigma(T_i)$, then q is incremented so...

$q=\sigma(T_i)$