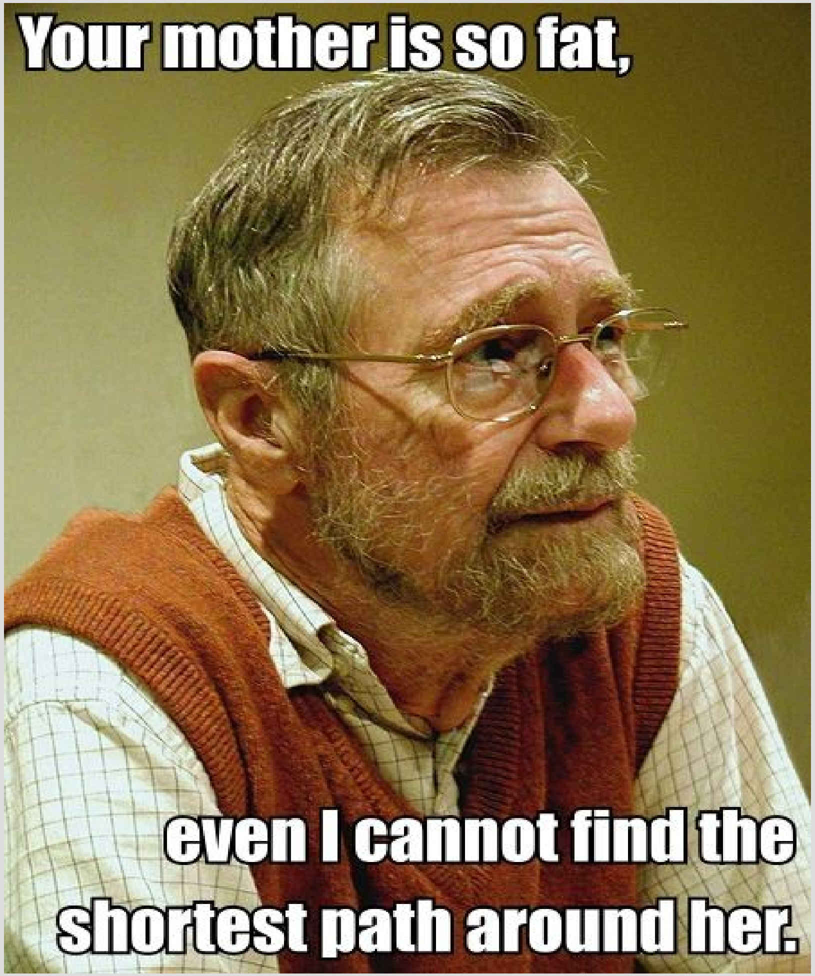
# Weighted graphs



# Weighted graph

Edges in weighted graph are assigned a weight: $w(v_1, v_2)$, where $v_1, v_2$ in V

If path $p = \langle v_0, v_1, \ldots v_k \rangle$ then the weight is: $w(p) = \sum^{k}_{i=1}(v_{i-1},v_i)$

Shortest Path:

$\delta(u,v)$: $\min\{w(p) : v_0=u, v_k=v)\}$

# Shortest paths

Today we will look at <u>single-source shorted paths</u>

This finds the shortest path from some starting vertex, s, to any other vertex on the graph (if it exists)

This creates $G_\pi$, the shortest path tree

# Shortest paths

Optimal substructure: Let $\delta(v_0,v_k)=p$, then for all $0 \le i \le j \le k$, $\delta(v_i,v_j)=p_{i,j}= \langle v_i, v_{i+1}, \ldots v_j \rangle$

Proof?

Where have we seen this before?

# Shortest paths

Optimal substructure: Let $\delta(v_0,v_k)=p$, then for all $0 \leq i \leq j \leq k$, $\delta(v_i,v_j)=p_{i,j}= \langle v_i, v_{i+1}, \ldots v_j\rangle$

Proof? Contradiction!

Suppose $w(p'_{i,j}) < p_{i,j}$, then let $p'_{0,k} = p_{0,i}\ p'_{i,j}\ p_{j,k}$ then $w(p'_{0,k}) < w(p)$

# Shortest path

We will do the same thing we have done before with BFS and DFS:

Makes a queue and put in/pull out

Two major differences:

(1) How to remove from queue (min)

(2) Update “grey” vertexes (“relax”)

# Relaxation

We will only do <u>relaxation</u> on the values v.d (min weight) for vertex v

Relax(u,v,w) (i.e. min() function)

```
if(v.d > u.d + w(u,v))
   v.d = u.d+w(u,v)
   v.π=u
```

# Relaxation

We will assume all vertices start with $v.d=\infty, v.\pi=NIL$ except s, $s.d=0$

This will take $O(|V|)$ time

This will not effect the asymptotic runtime as it will be at least $O(|V|)$ to find single-source shortest path

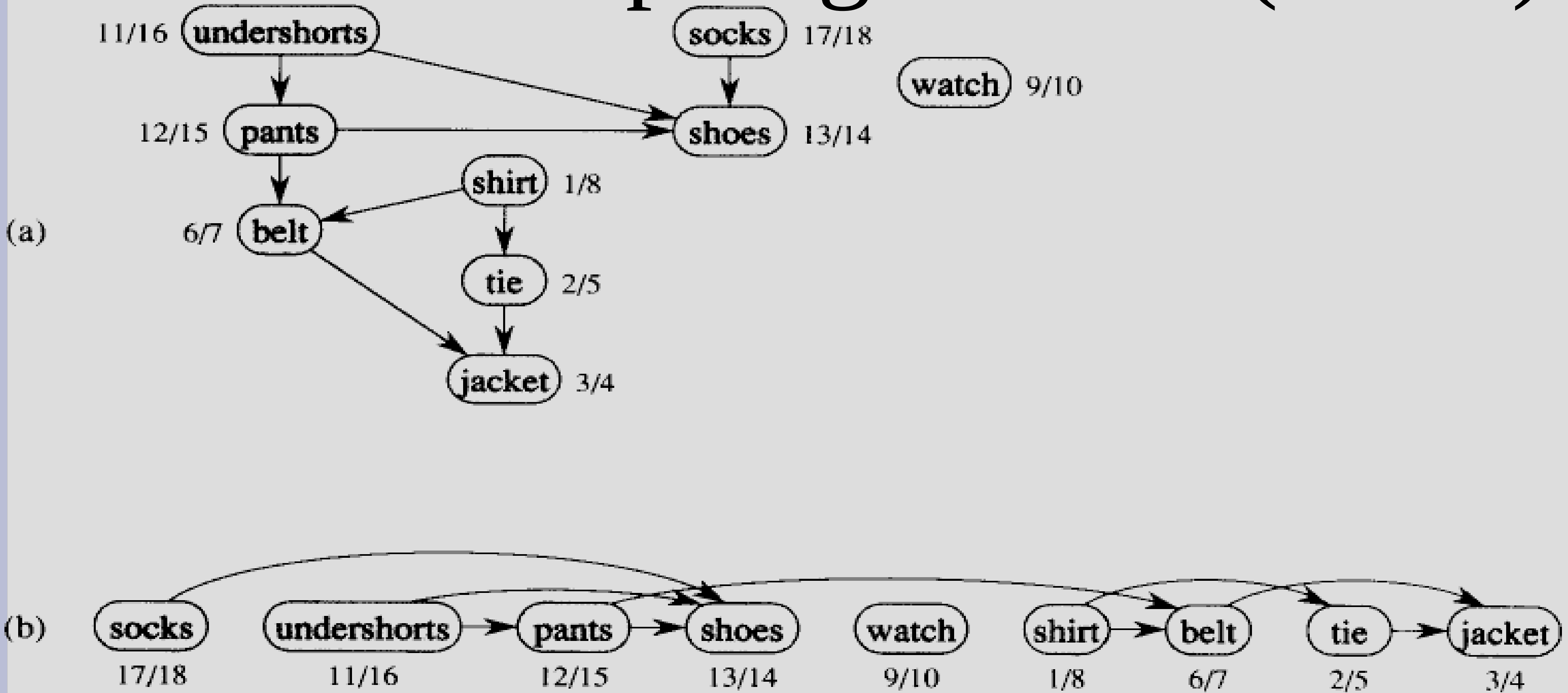
# Relaxation

Relaxation properties:

1. $\delta(s,v) \leq \delta(s,u) + \delta(u,v)$ (triangle inequality)
2. $v.d \geq \delta(s,v)$, v.d is monotonically decreasing
3. if no path, $v.d = \delta(s,v) = \infty$
4. if $\delta(s,v)$, when $(v.\pi).d = \delta(s, v.\pi)$ then relax($v.\pi, v, w$) causes $v.d = \delta(s,v)$
5. if $\delta(v_0, v_k) = p_{0,k}$, then when relaxed in order $(v_0, v_1), (v_1, v_2), \ldots (v_{k-1}, v_k)$ then $v_k.d = \delta(v_0, v_k)$ even if other relax happen
6. when $v.d = \delta(s,v)$ for all v in V, $G_\pi$ is shortest path tree rooted at s

# Directed Acyclic Graphs

DFS can do topological sort (DAG)

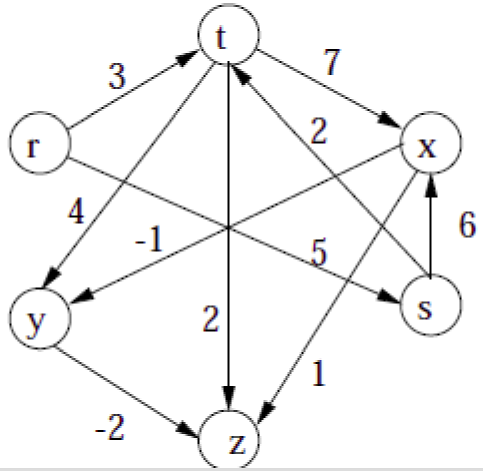



Run DFS, sort in decreasing finish time

# Directed Acyclic Graphs

```
DAG-shortest-paths(G,w,s)
topologically sort G
initialize graph from s
for each u in V in topological order
   for each v in G.Adj[u]
      Relax(u,v,w)
```

Runtime: $O(|V| + |E|)$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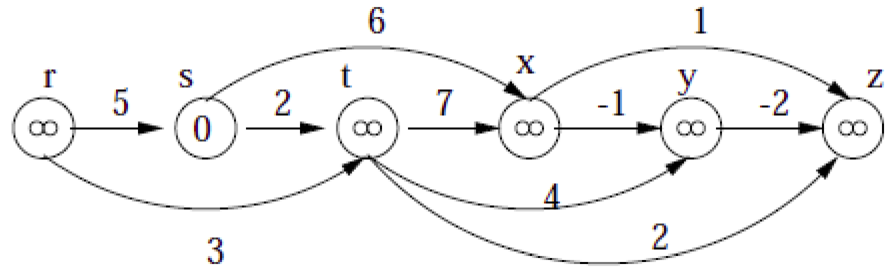
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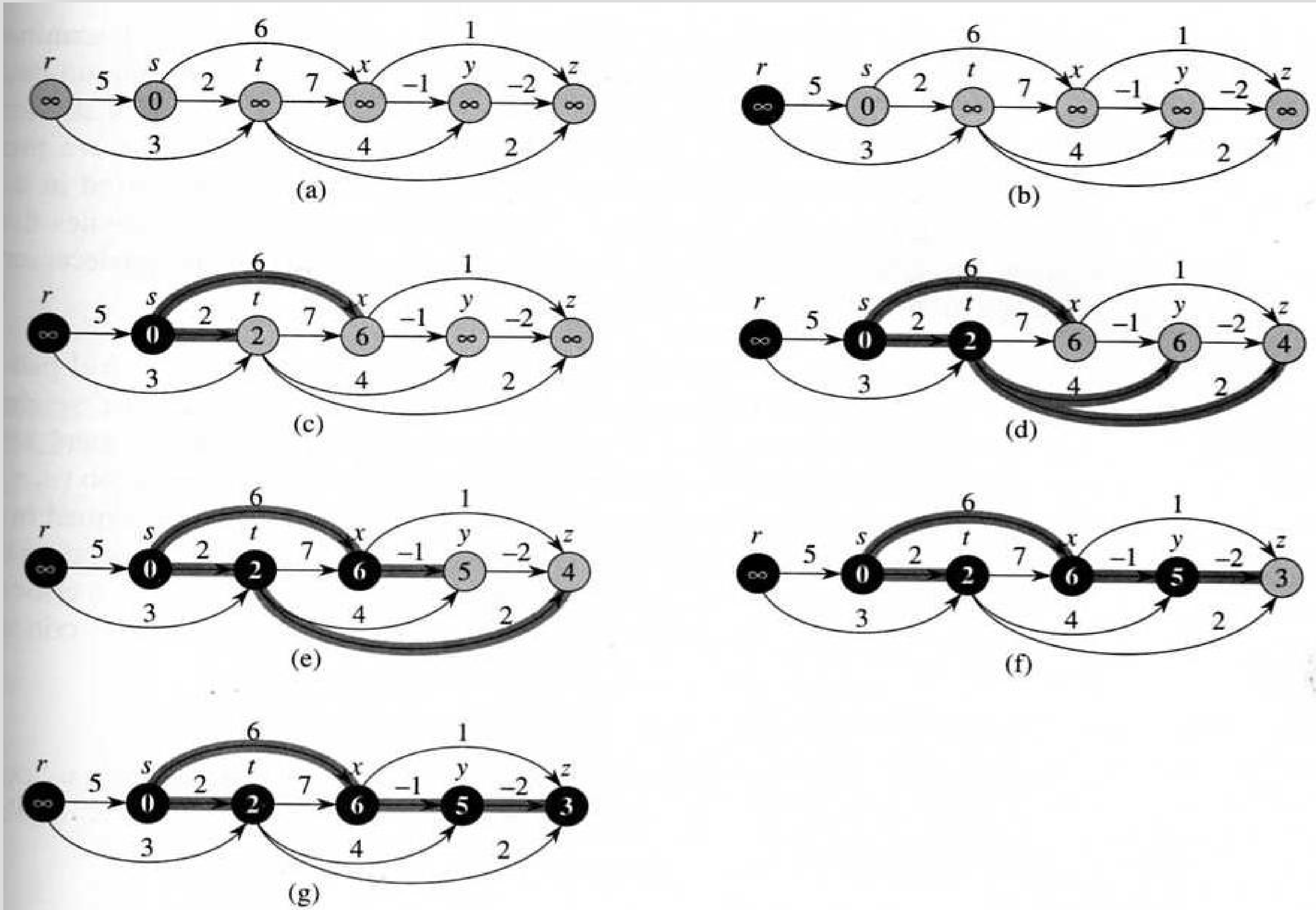

(a)
(b)
(c)
(d)
(e)
(f)
(g)

# Directed Acyclic Graphs

Correctness:

Prove it!

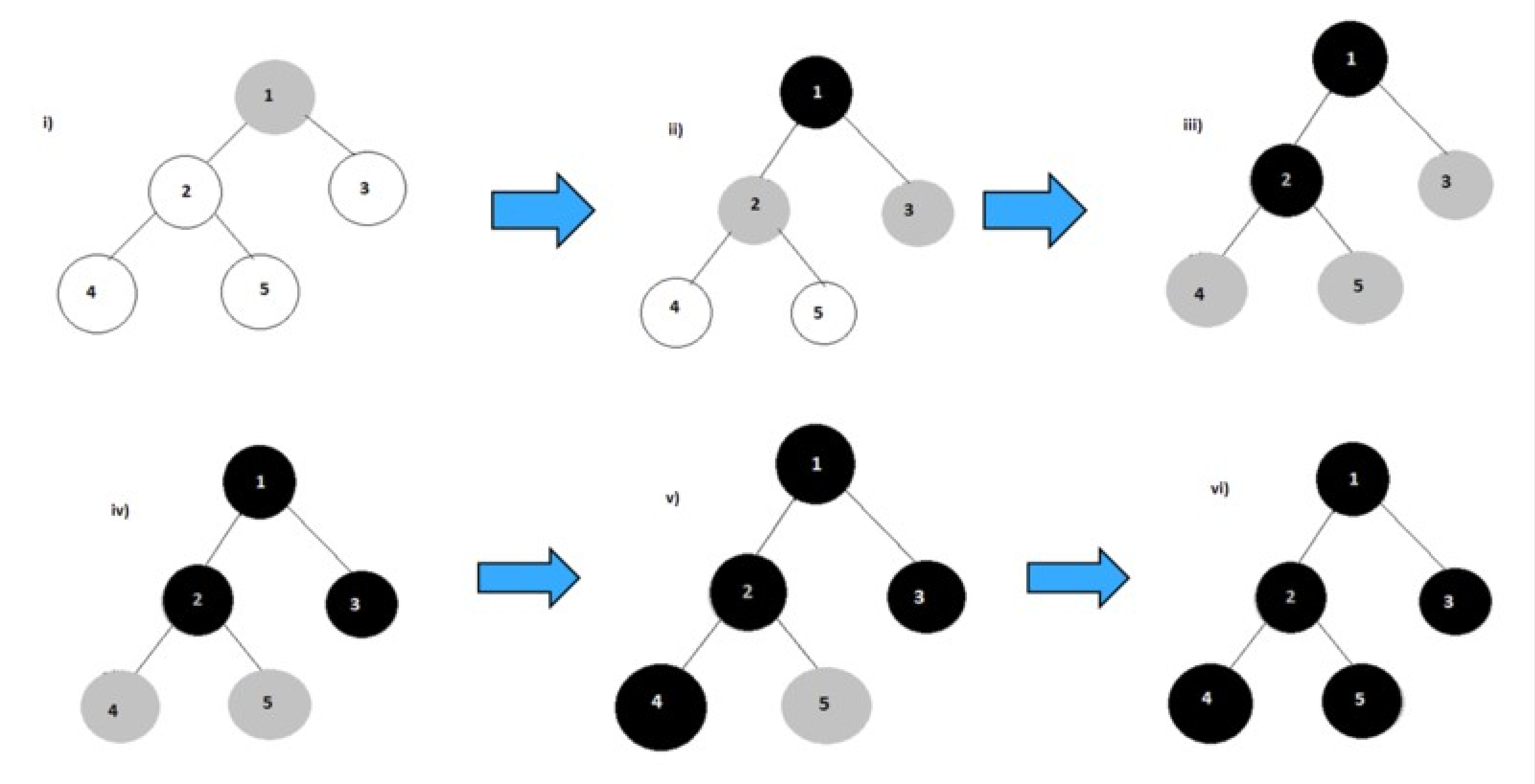
# Directed Acyclic Graphs

Correctness:
By definition of topological order,
When relaxing vertex v, we have already relaxed any preceding vertices

So by relaxation property 5, we have found the shortest path to all v

# BFS (unweighted graphs)

Create FIFO queue to explore unvisited nodes



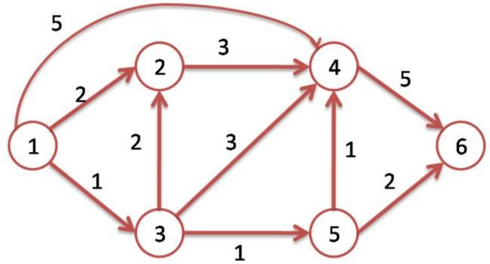
# Dijkstra

Dijkstra's algorithm is the BFS equivalent for non-negative weight graphs

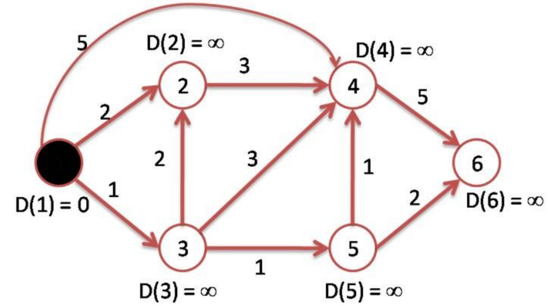
# Dijkstra

```
Dijkstra(G,w,s)
initialize G from s
Q = G.V, S = empty
while Q not empty
   u = Extract-min(Q)
   S = S U {u}
   for each v int G.Adj[u]
      relax(u,v,w)
```

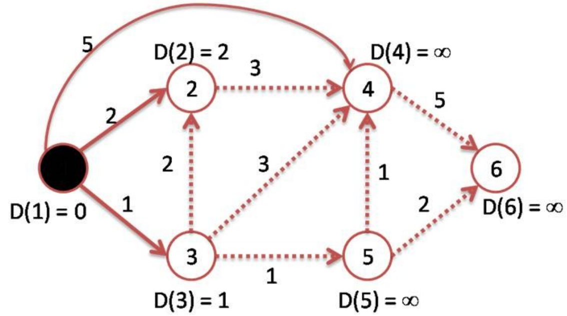
S optional



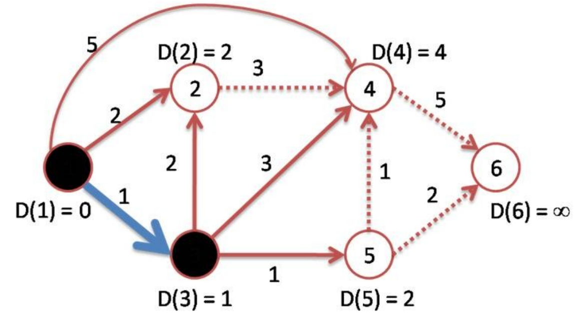
(a) Network Model



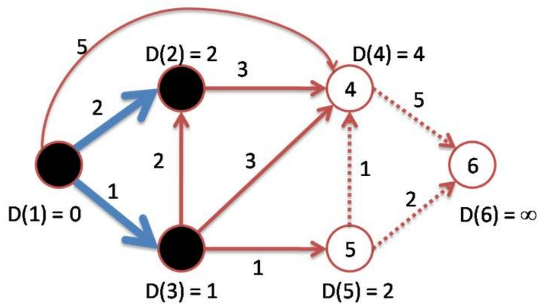

(b) Distance initialized



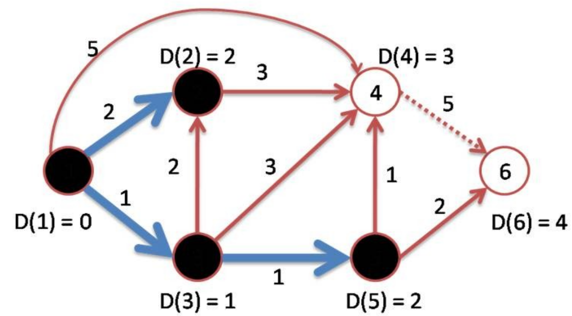

(c) Distance to adjacent nodes updated



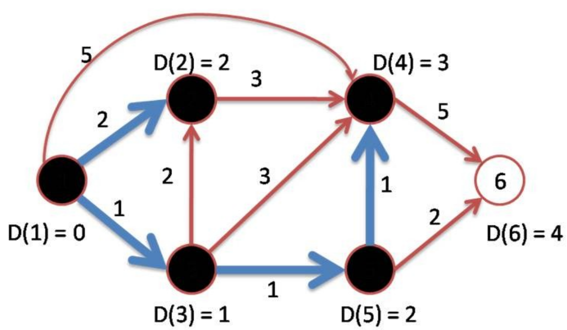

(d) Node 3 selected



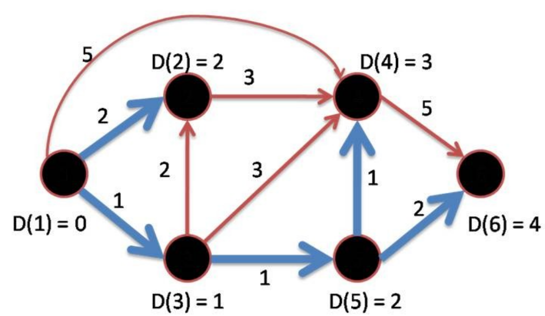

(e) Node 2 selected




(f) Node 5 selected




(g) Node 4 selected




(h) Shortest path found

# Dijkstra

Runtime?

# Dijkstra

Runtime:

Extract-min() run |V| times

Relax runs Decrease-key() |E| times

Both take O(lg n) time

So O( (|V| + |E|) lg |V|) time

(can get to O(|V|lg|V| + E) using Fibonacci heaps)

# Dijkstra

Runtime note:

If G is almost fully connected,

$|E| \approx |V|^2$

Use a simple array to store v.d

Extract-min() = $O(|V|)$

Decrease-key() = $O(1)$

total: $O(|V|^2 + E)$

# Dijkstra

Correctness: (p.660)
Sufficient to prove when u added to S, u.d = $\delta(s,u)$

Base: s added to S first, s.d=0=$\delta(s,s)$

Termination: Loop ends after Q is empty, so V=S and we done

# Dijkstra

Step: Assume v in S has $v.d = \delta(s,v)$
Let y be the first vertex outside S
on path of $\delta(s,u)$

We know by relaxation property 4,
that $\delta(s,y)=y.d$ (optimal sub-structure)

$y.d = \delta(s,y) \leq \delta(s,u) = u.d$, as $w(p)\geq 0$

# Dijkstra

Step: Assume v in S has v.d = $\delta(s,v)$

But as u was picked before y,

u.d $\leq$ y.d, combined with y.d $\leq$ u.d

y.d=u.d

Thus y.d = $\delta(s,y) = \delta(s,u)$ = u.d