1. Consider the system

\[ Ax \equiv \begin{pmatrix} -0.001 & 1.001 \\ 0.001 & -0.001 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \equiv b \]

whose solution is \( x_1 = x_2 = 1 \) and the system

\[ (A + \Delta A)y = b + \Delta b \]

where \( \Delta A = \varepsilon |A|, \Delta b = \varepsilon |b| \). In the following we let \( \varepsilon = 10^{-4} \).

Compute \( \kappa_\infty(A) \) (matlab OK). Compute the actual value of \( \|x - y\|_\infty / \|x\|_\infty \) and its estimate obtained from using the (standard) condition number \( \kappa_\infty \) (Theorem 2 in notes).

2. (a) Show that the following matrix is singular

\[ A = \begin{pmatrix} 1 & 2 & -1 \\ 2 & 1 & 1 \\ -1 & 1 & -2 \end{pmatrix} \]

(b) What is the range of \( A \)? What is its null space?

(c) Consider the matrix \( B \) obtained from \( A \) by adding \( \eta = 0.001 \) to the entry \((1,3)\) (So \( B = A + \eta e_1 e_3^T \)). Without computing the inverse of \( B \), show that \( \|B^{-1}\|_1 \geq 3,000 \).

(d) Find a lower bound for the condition number \( \kappa_1(B) \).

3. Consider the \( n \times n \) matrix

\[ A_n = \begin{pmatrix} 1 & -2 & 1 & \ldots & \ldots & -2 & 1 \\ -2 & 1 & \ddots & \ddots & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ -2 & 1 & \ddots & \ddots & \ddots & \ddots & \ddots \\ 1 & -2 & \ldots & \ddots & \ddots & \ddots & \ddots \\ \end{pmatrix} \]

What is \( A_n^{-1} \)? [Hint: Write \( A_n = I - E_n \) and use power expansion for the inverse ...]

Calculate the condition numbers \( \kappa_1(A_n) \) and \( \kappa_\infty(A_n) \). Verify your results with matlab for the case \( n = 10 \).

4. Consider the matrices

\[ A = \begin{pmatrix} 1 & -1 \\ 1 & -1.00001 \end{pmatrix} \quad B = \begin{pmatrix} 1 & -1 \\ -1 & 1.00001 \end{pmatrix} \]

What is ratio of the largest to smallest eigenvalues (in modulus) for \( A \) and for \( B \)? Show that \( \kappa_2(A) = \kappa_2(B) \). What can you conclude about the ratio of the largest to smallest eigenvalues as a way of estimating sensitivity of a linear system? Would you consider \( A \) to be well-conditioned or ill-conditioned?
5. (a) Apply the matlab script you developed in Homework 2 to obtain LU factorization of the following $n \times n$ matrix:

$$A = \begin{pmatrix}
1 & 0 & \cdots & \cdots & 0 & 1 \\
-1 & 1 & 0 & \cdots & 0 & 1 \\
-1 & -1 & 1 & 0 & \cdots & \vdots \\
\vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\
-1 & -1 & \cdots & \cdots & -1 & 1
\end{pmatrix}$$

[Hint: As example take an $8 \times 8$ matrix then give the L, U factors for a general $n \times n$ matrix. Use your own matlab script, not the matlab function.]

(b) What can you say about $\|U\|_{\infty}$ as a function of the size $n$? How do you relate this result to question 2 of Practice Exercises set # 3.

(c) Compute, for the case $n = 8$, the matrix $128 \ast A^{-1}$ (in matlab). Can you tell what the inverse of $A$ is in general (for any $n$)? Prove your result. What is (exactly) the 1-norm condition number of $A$ as a function of $n$. Any comments?

6. Show that the following symmetric matrices are not positive definite [without computing eigenvalues]

$$A = \begin{pmatrix}
0 & 3 & 1 \\
3 & 2 & 0 \\
1 & 0 & 2
\end{pmatrix} \quad B = \begin{pmatrix}
2 & 3 & 1 \\
3 & 2 & 0 \\
1 & 0 & 2
\end{pmatrix} \quad C = \begin{pmatrix}
1 & 0 & 2 \\
0 & 2 & 0 \\
2 & 0 & 2
\end{pmatrix}$$

7. Consider the matrix

$$A = \begin{pmatrix}
4 & -2 & 4 \\
-2 & 2 & -2 \\
4 & -2 & 8
\end{pmatrix}$$

(a) Find the LU factorization of $A$ (no pivoting);

(b) Find its LDLT factorization from the previous question;

(c) Find its Cholesky factorization (again using (a)).

(d) Suppose you want to decrease $a_{33}$ from its value of 8, so that the modified $A$ still admits a Cholesky factorization. What is the lower limit for $a_{33}$ (also say if the limit is itself acceptable).