1. You will find in the class web-site [see matlab page], a sample set of seasonal farm employment data \((t_i, y_i)\) over about an 18 month period, where \(t_i\) represents months, and \(y_i\) is the employed population in millions (file \(\text{hw4_q1.dat}\)). It is thought that this population, \(y(t)\), evolves with time according to a function of the form:

\[
y(t) = a_1 + a_2t + a_3 \cos t.
\]

Find \(a_1, a_2, a_3\) by least-squares data fitting. Then plot the function you find. On the same plot show also the observed population [using a square for each point].

2. The purpose of this exercise is to test 3 different ways of computing the QR factorization of a matrix

(a) The classical Gram-Schmidt algorithm

(b) The modified Gram-Schmidt algorithm

(c) The Cholesky factorization of \(A^T A\)

Explain how the Cholesky factorization of \(A^T A\) can be used. In the following you should use the script \(\text{cho1R}\) that is posted (not the \(\text{chol}\) function from matlab). You can use \(\text{inv}\) to invert triangular matrices.

A data set is posted on the class web-site (see the matlab page). Write a script which loads the matrix and then for each of the three methods above compute the \(Q\) anmd \(R\) factors and the error measures

\[
\|A - Q \ast R\|_2, \quad \|I - Q^T \ast Q\|_2
\]

Present your result int the form of a table and comment on them.

3. Find the matrix \(X\) that is closest to the matrix \(A\), in the Frobenius norm sense, among all matrices of the form \(\alpha B + \beta C + \gamma D\), when :

\[
A = \begin{pmatrix} 1 & 3 & -1 \\ 2 & -1 & 3 \end{pmatrix}; \quad B = \begin{pmatrix} 0 & 1 & -1 \\ 1 & 0 & 1 \end{pmatrix}; \quad C = \begin{pmatrix} -2 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}; \quad D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.
\]

4. Suppose that for a given matrix \(M \in \mathbb{R}^{m \times n}\) you obtain a QR factorization which you write in the form:

\[
M = Q \times \begin{pmatrix} A & B \\ C \end{pmatrix}
\]

where \(A\) is \(k \times k\) invertible, \(Q \in \mathbb{R}^{m \times m}\) is orthogonal \((Q^T Q = I)\) and \(C \in \mathbb{R}^{(m-k) \times (n-k)}\). Show that \(\det(M^T M) = \det(A^T A) \cdot \det(C^T C)\)

5. Consider the matrix

\[
A(\varepsilon) = \begin{pmatrix} 1 & 1 & 1 & 1 \\ \varepsilon & 0 & 0 & 0 \\ 0 & \varepsilon & 0 & 0 \\ 0 & 0 & \varepsilon & 0 \\ 0 & 0 & 0 & \varepsilon \end{pmatrix}
\]

where \(\varepsilon\) is a small number such that \(\text{fl}(1 + \varepsilon^2) = 1\).
(a) What are the $Q$ matrices that you would obtain by using the classical and the modified Gram-Schmidt algorithms?

(b) Compare the orthogonality of the two $Q$ factors obtained from the two algorithms (as was done in Question 2).

(c) Assume that you need to solve a least-squares problem with the matrix $A$, using the same floating point arithmetic (i.e., we still have $fl(1 + \varepsilon^2) = 1$). What happens if you use the simplistic approach of the normal equations (e.g. solving $A^T A x = A^T b$ via Gaussian elimination)?

6. Implement a matlab function which computes the Householder QR factorization of a full-rank matrix $A$. [formulations which produce a negative diagonal element in $R$ are OK]. The function should be as follows:

$$[V, \text{bet}] = \text{housQR}(A)$$

where $V$ is the matrix of the vectors $v_1, \ldots, v_m$ are related to the successive Householder reflectors $P_j = I - \beta_j v_j v_j^T$ used to transform $A$ into upper triangular form and bet is the vector of the coefficients $\beta_j$. Show the matlab function.

7. Test the program developed above for the same data as the one used for Question 2 above. Obtain the $Q_1, R_1$ matrices of the factorization $A = Q_1 R_1$ – where $Q_1$ is $n \times m$ and $R_1$ is $m \times m$, from the Householder factorization. Compare the $R_1$ matrix obtained from the Householder method with the $R$ matrix obtained from the Modified Gram-Schmidt method seen in Question 2. (show the 1-norm of $R_1 - R$). Similarly, compute the norms $\|I - Q^T Q\|_1$ and $\|I - Q_1^T Q_1\|_1$. 