ERROR AND SENSITIVITY ANALYSIS FOR SYSTEMS
OF LINEAR EQUATIONS

- Conditioning of linear systems.
- Estimating errors for solutions of linear systems
- Backward error analysis
- Relative element-wise error analysis
Perturbation analysis for linear systems \((Ax = b)\)

Question addressed by perturbation analysis: determine the variation of the solution \(x\) when the data, namely \(A\) and \(b\), undergoes small variations. Problem is **Ill-conditioned** if small variations in data cause very large variation in the solution.
Let \( E \), be an \( n \times n \) matrix and \( e_b \) be an \( n \)-vector.

“Perturb” \( A \) into \( A(\epsilon) = A + \epsilon E \) and \( b \) into \( b + \epsilon e_b \).

Note: \( A + \epsilon E \) is nonsingular for \( \epsilon \) small enough.

Why?

The solution \( x(\epsilon) \) of the perturbed system is s.t.

\[
(A + \epsilon E)x(\epsilon) = b + \epsilon e_b.
\]

Let \( \delta(\epsilon) = x(\epsilon) - x \). Then,

\[
(A + \epsilon E)\delta(\epsilon) = (b + \epsilon e_b) - (A + \epsilon E)x = \epsilon (e_b - Ex)
\]

\[
\delta(\epsilon) = \epsilon (A + \epsilon E)^{-1}(e_b - Ex).
\]
\( x(\epsilon) \) is differentiable at \( \epsilon = 0 \) and its derivative is

\[
x'(0) = \lim_{\epsilon \to 0} \frac{\delta(\epsilon)}{\epsilon} = A^{-1}(e_b - Ex) .
\]

A small variation \([\epsilon E, \epsilon e_b]\) will cause the solution to vary by roughly \( \epsilon x'(0) = \epsilon A^{-1}(e_b - Ex) \).

The relative variation is such that

\[
\frac{\|x(\epsilon) - x\|}{\|x\|} \leq \epsilon \|A^{-1}\| \left( \frac{\|e_b\|}{\|x\|} + \|E\| \right) + O(\epsilon^2)
\]

Since \( \|b\| \leq \|A\|\|x\| \):

\[
\frac{\|x(\epsilon) - x\|}{\|x\|} \leq \epsilon \|A\| \|A^{-1}\| \left( \frac{\|e_b\|}{\|b\|} + \frac{\|E\|}{\|A\|} \right) + O(\epsilon^2)
\]
The quantity $\kappa(A) = \| A \| \| A^{-1} \|$ is called the condition number of the linear system with respect to the norm $\| \cdot \|$. When using the $p$-norms we write:

$$\kappa_p(A) = \| A \|_p \| A^{-1} \|_p$$

Note: $\kappa_2(A) = \sigma_{max}(A)/\sigma_{min}(A)$ = ratio of largest to smallest singular values of $A$. Allows to define $\kappa_2(A)$ when $A$ is not square.

Determinant *is not* a good indication of sensitivity

Small eigenvalues *do not* always give a good indication of poor conditioning.
Consider, for a large $\alpha$, the $n \times n$ matrix

$$A = I + \alpha e_1 e_n^T$$

Inverse of $A$ is:

$$A^{-1} = I - \alpha e_1 e_n^T$$

For the $\infty$-norm we have

$$\|A\|_{\infty} = \|A^{-1}\|_{\infty} = 1 + |\alpha|$$

so that

$$\kappa_\infty(A) = (1 + |\alpha|)^2.$$  

Can give a very large condition number for a large $\alpha$ – but all the eigenvalues of $A$ are equal to one.
Rigorous norm-based error bounds

Previous bound is valid only when perturbation is “small enough,” where “small” is not precisely defined.

New bound valid within an explicitly given neighborhood.

THEOREM 1: Assume that \((A + E)y = b + e_b\) and \(Ax = b\) and that \(\|A^{-1}\|\|E\| < 1\). Then \(A + E\) is nonsingular and

\[
\frac{\|x - y\|}{\|x\|} \leq \frac{\|A^{-1}\| \|A\|}{1 - \|A^{-1}\| \|E\|} \left( \frac{\|E\|}{\|A\|} + \frac{\|e_b\|}{\|b\|} \right)
\]

To prove, first need to show that \(A + E\) is nonsingular if \(A\) is nonsingular and \(E\) is small.
Begin with simple case:

**LEMMA:** If $\|E\| < 1$ then $I - E$ is nonsingular and

$$\|(I - E)^{-1}\| \leq \frac{1}{1 - \|E\|}$$

*Proof* is based on following 5 steps

a) Show: If $\|E\| < 1$ then $I - E$ is nonsingular

b) Show: $(I - E)(I + E + E^2 + \cdots + E^k) = I - E^{k+1}$.

c) From which we get:

$$(I - E)^{-1} = \sum_{i=0}^{k} E^i + (I - E)^{-1} E^{k+1} \rightarrow$$
d) \((I - E)^{-1} = \lim_{k \to \infty} \sum_{i=0}^{k} E^i\). We write this as

\[(I - E)^{-1} = \sum_{i=0}^{\infty} E^i\]

e) Finally:

\[\| (I - E)^{-1} \| = \left\| \lim_{k \to \infty} \sum_{i=0}^{k} E^i \right\| = \lim_{k \to \infty} \left\| \sum_{i=0}^{k} E^i \right\| \leq \lim_{k \to \infty} \sum_{i=0}^{k} \| E^i \| \leq \lim_{k \to \infty} \sum_{i=0}^{k} \| E \|^i \leq \frac{1}{1 - \| E \|}\]
Can generalize result:

**LEMMA:** If $A$ is nonsingular and $\|A^{-1}\| \|E\| < 1$ then $A + E$ is non-singular and

$$\|(A + E)^{-1}\| \leq \frac{\|A^{-1}\|}{1 - \|A^{-1}\| \|E\|}$$

Proof is based on relation $A + E = A(I + A^{-1}E)$ and use of previous lemma.

Now we can prove the theorem:

**THEOREM 1:** Assume that $(A + E)y = b + e_b$ and $Ax = b$ and that $\|A^{-1}\|\|E\| < 1$. Then $A + E$ is nonsingular and

$$\frac{\|x - y\|}{\|x\|} \leq \frac{\|A^{-1}\| \|A\|}{1 - \|A^{-1}\| \|E\|} \left(\frac{\|E\|}{\|A\|} + \frac{\|e_b\|}{\|b\|}\right)$$
Proof: From $(A + E)y = b + e_b$ and $Ax = b$ we get

$$(A + E)(y - x) = e_b - Ex.$$ Hence:

$$y - x = (A + E)^{-1}(e_b - Ex)$$

Taking norms $\rightarrow \|y - x\| \leq \|(A + E)^{-1}\| \left(\|e_b\| + \|E\|\|x\|\right)$

Dividing by $\|x\|$ and using result of lemma

$$\frac{\|y - x\|}{\|x\|} \leq \|(A + E)^{-1}\| \left(\|e_b\|/\|x\| + \|E\|\right)$$

$$\leq \frac{\|A^{-1}\|}{1 - \|A^{-1}\|\|E\|} \left(\|e_b\|/\|x\| + \|E\|\right)$$

$$\leq \frac{\|A^{-1}\|\|A\|}{1 - \|A^{-1}\|\|E\|} \left(\|e_b\|/\|A\|\|x\| + \|E\|/\|A\|\right)$$

Result follows by using inequality $\|A\|\|x\| \geq \|b\|$.... QED
Simplification when $e_b = 0$:

\[
\frac{\|x - y\|}{\|x\|} \leq \frac{\|A^{-1}\| \|E\|}{1 - \|A^{-1}\| \|E\|}
\]

Simplification when $E = 0$:

\[
\frac{\|x - y\|}{\|x\|} \leq \frac{\|A^{-1}\| \|A\| \|e_b\|}{\|b\|}
\]

Slightly less general form: Assume that $\|E\|/\|A\| \leq \delta$ and $\|e_b\|/\|b\| \leq \delta$ and $\delta \kappa(A) < 1$ then

\[
\frac{\|x - y\|}{\|x\|} \leq \frac{2\delta \kappa(A)}{1 - \delta \kappa(A)}
\]

Show the above result
THEOREM 2: Let \((A + \Delta A)y = b + \Delta b\) and \(Ax = b\) where \(\|\Delta A\| \leq \epsilon \|E\|\), \(\|\Delta b\| \leq \epsilon \|e_b\|\), and assume that \(\epsilon \|A^{-1}\| \|E\| < 1\). Then

\[
\frac{\|x - y\|}{\|x\|} \leq \frac{\epsilon \|A^{-1}\| \|A\|}{1 - \epsilon \|A^{-1}\| \|E\|} \left(\frac{\|e_b\|}{\|b\|} + \frac{\|E\|}{\|A\|}\right)
\]

Results to be seen later are of this type.
**Normwise backward error**

We solve $Ax = b$ and find an approximate solution $y$

**Question:** Find smallest perturbation to apply to $A, b$ so that *exact* solution of perturbed system is $y$
Normwise backward error in just $A$ or $b$

Suppose we model entire perturbation in RHS $b$.

- Let $r = b - Ay$ be the residual. Then $y$ satisfies $Ay = b + \Delta b$ with $\Delta b = -r$ exactly.

- The relative perturbation to the RHS is $\frac{\|r\|}{\|b\|}$.

Suppose we model entire perturbation in matrix $A$.

- Then $y$ satisfies $\left( A + \frac{ry^T}{y^Ty} \right) y = b$

- The relative perturbation to the matrix is

$$\frac{\|ry^T\|_2}{\|y^Ty\|_2} / \|A\|_2 = \frac{\|r\|_2}{\|A\| \|y\|_2}$$
Normwise backward error in both $A$ & $b$

For a given $y$ and given perturbation directions $E$, $e_b$, we define the Normwise backward error:

$$\eta_{E,e_b}(y) = \min\{\epsilon \mid (A + \Delta A)y = b + \Delta b;$$
for all $\Delta A, \Delta b$ satisfying: $\|\Delta A\| \leq \epsilon \|E\|$;
and $\|\Delta b\| \leq \epsilon \|e_b\|\}$$

In other words $\eta_{E,e_b}(y)$ is the smallest $\epsilon$ for which

\[
(1) \quad \begin{cases} 
(A + \Delta A)y = b + \Delta b; \\
\|\Delta A\| \leq \epsilon \|E\|; \\
\|\Delta b\| \leq \epsilon \|e_b\|
\end{cases}
\]
$y$ is given (a computed solution). $E$ and $e_b$ to be selected (most likely 'directions of perturbation for $A$ and $b$').

Typical choice: $E = A$, $e_b = b$

Explain why this is not unreasonable

Let $r = b - Ay$. Then we have:

**THEOREM 3:** \[ \eta_{E,e_b}(y) = \frac{\|r\|}{\|E\|\|y\| + \|e_b\|} \]

Normwise backward error is for case $E = A$, $e_b = b$:

\[ \eta_{A,b}(y) = \frac{\|r\|}{\|A\|\|y\| + \|b\|} \]
Show how this can be used in practice as a means to stop some iterative method which computes a sequence of approximate solutions to $Ax = b$.

Consider the $6 \times 6$ Vandermonde system $Ax = b$ where $a_{ij} = j^{2(i-1)}$, $b = A \ast [1, 1, \cdots, 1]^T$. We perturb $A$ by $E$, with $|E| \leq 10^{-10}|A|$ and $b$ similarly and solve the system. Evaluate the backward error for this case. Evaluate the forward bound provided by Theorem 2. Comment on the results.
Proof of Theorem 3

Let $D \equiv \|E\|\|y\| + \|e_b\|$ and $\eta \equiv \eta_{E,e_b}(y)$. The theorem states that $\eta = \|r\|/D$. Proof in 2 steps.

First: Any $\Delta A, \Delta b$ pair satisfying (1) is such that $\epsilon \geq \|r\|/D$. Indeed from (1) we have (recall that $r = b - Ay$)

$$Ay + \Delta Ay = b + \Delta b \rightarrow r = \Delta Ay - \Delta b \rightarrow$$

$$\|r\| \leq \|\Delta A\|\|y\| + \|\Delta b\| \leq \epsilon(\|E\|\|y\| + \|e_b\|) \rightarrow \epsilon \geq \frac{\|r\|}{D}$$

Second: We need to show an instance where the minimum value of $\|r\|/D$ is reached. Take the pair $\Delta A, \Delta b$:

$$\Delta A = \alpha rz^T; \quad \Delta b = \beta r \quad \text{with} \quad \alpha = \frac{\|E\|\|y\|}{D}; \quad \beta = \frac{\|e_b\|}{D}$$
The vector $z$ depends on the norm used - for the 2-norm: $z = y/\|y\|^2$. Here: Proof only for 2-norm

a) We need to verify that first part of (1) is satisfied:

$$(A + \Delta A)y = Ay + \alpha r \frac{y^T}{\|y\|^2} y = b - r + \alpha r$$

$$= b - (1 - \alpha)r = b - \left(1 - \frac{\|E\|\|y\|}{\|E\|\|y\| + \|e_b\|}\right)r$$

$$= b - \frac{\|e_b\|}{D}r = b + \beta r \quad \rightarrow$$

$$(A + \Delta A)y = b + \Delta b \quad \leftarrow \text{The desired result}$$
b) Finally: Must now verify that $\|\Delta A\| = \eta \|E\|$ and $\|\Delta b\| = \eta \|e_b\|$. Exercise: Show that $\|uv^T\|_2 = \|u\|_2 \|v\|_2$

$$\|\Delta A\| = \frac{|\alpha|}{\|y\|^2} \|ry^T\| = \frac{\|E\| \|y\| \|r\| \|y\|}{D \|y\|^2} = \eta \|E\|$$

$$\|\Delta b\| = \|\beta\| \|r\| = \frac{\|e_b\|}{D} \|r\| = \eta \|e_b\| \quad QED$$