Estimating condition numbers.

- Often we just want to get a lower bound for condition number (it is 'worse than ...').
- We want to estimate \( \|A\| \|A^{-1}\| \).
- The norm \( \|A\| \) is usually easy to compute but \( \|A^{-1}\| \) is not.
- We want: Avoid the expense of computing \( A^{-1} \) explicitly.

**Idea:**

- Select a vector \( v \) so that \( \|v\| = 1 \) but \( \|Av\| = \tau \) is small.
- Then: \( \|A^{-1}\| \geq 1/\tau \) (show why) and:
  \[ \kappa(A) \geq \frac{\|A\|}{\tau} \]

Condition numbers and near-singularity

- \( 1/\kappa \approx \) relative distance to nearest singular matrix.

Let \( A, B \) be two \( n \times n \) matrices with \( A \) nonsingular and \( B \) singular. Then

\[ \frac{1}{\kappa(A)} \leq \frac{\|A - B\|}{\|A\|} \]

**Proof:** \( B \) singular \( \implies \exists x \neq 0 \) such that \( Bx = 0 \).

\[ \|x\| = \|A^{-1}Ax\| \leq \|A^{-1}\| \|Ax\| = \|A^{-1}\| \|(A - B)x\| \leq \|A^{-1}\| \|A - B\| \|x\| \]

Divide both sides by \( \|x\| \times \kappa(A) = \|x\| \|A\| \|A^{-1}\| \implies \text{result. QED.} \]

Example:

Let \( A = \begin{pmatrix} 1 & 1 \\ 1 & 0.99 \end{pmatrix} \) and \( B = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \)

Then \( \frac{1}{\kappa_1(A)} \leq \frac{0.01}{2} \implies \kappa_1(A) \geq \frac{2}{0.01} = 200. \)

- It can be shown that (Kahan)

\[ \frac{1}{\kappa(A)} = \min_B \left\{ \frac{\|A - B\|}{\|A\|} \mid \det(B) = 0 \right\} \]
Estimating errors from residual norms

Let \( \tilde{x} \) an approximate solution to system \( Ax = b \) (e.g., computed from an iterative process). We can compute the residual norm:

\[
\|r\| = \|b - A\tilde{x}\|
\]

Question: How to estimate the error \( \|x - \tilde{x}\| \) from \( \|r\| \)?

- One option is to use the inequality
  \[
  \frac{\|x - \tilde{x}\|}{\|x\|} \leq \kappa(A) \frac{\|r\|}{\|b\|}.
  \]

- We must have an estimate of \( \kappa(A) \).

Proof of inequality.

First, note that \( A(x - \tilde{x}) = b - A\tilde{x} = r \). So:

\[
\|x - \tilde{x}\| = \|A^{-1}r\| \leq \|A^{-1}\| \|r\|
\]

Also note that from the relation \( b = Ax \), we get

\[
\|b\| = \|Ax\| \leq \|A\| \|x\| \implies \|x\| \geq \frac{\|b\|}{\|A\|}
\]

Therefore,

\[
\frac{\|x - \tilde{x}\|}{\|x\|} \leq \frac{\|A^{-1}\| \|r\|}{\|b\| / \|A\|} = \kappa(A) \frac{\|r\|}{\|b\|}
\]

Show that

\[
\frac{\|x - \tilde{x}\|}{\|x\|} \geq \frac{1}{\kappa(A)} \frac{\|r\|}{\|b\|}.
\]