1 Quiz 7

1. Let \( f : A \to B \) and define the relation \( \equiv \) on \( A \) by \( x \equiv y \) iff \( f(x) = f(y) \). Prove that \( \equiv \) is an equivalence relation.

Solution:
(1) Show the relation is reflexive. Let \( x \in A \). Since \( f(x) = f(x) \), \( x \equiv x \). We know the relation \( \equiv \) is reflexive.
(2) Show the relation is symmetric. Let \( x, y \in A \) and \( x \equiv y \). Since \( f(x) = f(y) \), that is \( f(y) = f(x) \), \( y \equiv x \). So the relation \( \equiv \) is symmetric.
(3) Show the relation is transitive. Let \( x, y, z \in A \), \( x \equiv y \), and \( y \equiv z \). Since \( f(x) = f(y) \) and \( f(y) = f(z) \), \( f(x) = f(z) \). Therefore, \( x \equiv z \). The relation is transitive.

Hence, \( \equiv \) is an equivalence relation.

2. Let \( | \) be the divisibility relation on the positive integers. Prove that if \( a | b \) and \( b | a \) then \( a = b \).

Solution:
Because \( a | b \) and \( b | a \), \( b = ma \), \( a = nb. \) (\( m, n \in \mathbb{N} \))

\[ b = ma = mnb, \]

so \( mn = 1 \) and \( m, n \) are positive integers. So we know \( m = 1 \) and \( n = 1 \). Therefore, \( a = b \).

2 Integer representations

**Theorem:** Let \( b \geq 2 \) be an integer. Then every positive integer \( n \) has a unique representation in base \( b \) as

\[ n = a_k b^k + a_{k-1} b^{k-1} + \ldots + a_1 b^1 + a_0 b^0 \]

Where \( k \) is a nonnegative integer, \( a_0, a_1, \ldots a_k \) are nonegative integers less than \( b \) and \( a_k \neq 0 \).

The common bases include:
- **Binary:** base 2.
- **Octal:** base 8.
- **Decimal:** base 10.
- **Hexadecimal:** base 16.

**Example 1:** What is the octal expansion of the number \( (3940)_{10} \)? (convert decimal to other bases)

**solution:** Dividing 3940 by 8, we obtain a quotient of 492 and remainder of 4. Thus,

\[ 3940 = 492 \cdot 8 + 4. \]
Dividing 492 by 8, we obtain a quotient of 61 and a remainder of 4.

\[ 492 = 61 \cdot 8 + 4 \]

Hence, \(3940 = 492 \cdot 8 + 4 = (61 \cdot 8 + 4) \cdot 8 + 4 = 61 \cdot 8^2 + 4 \cdot 8 + 4 = (7 \cdot 8 + 5) \cdot 8^2 + 4 \cdot 8 + 4\]

\[ 3940 = 7 \cdot 8^3 + 5 \cdot 8^2 + 4 \cdot 8 + 4 \cdot 8^0 \]

Therefore, \((3940)_{10} = (7544)_8\).

**Example 2:** Determine the decimal expansions of the integers whose binary representations are given below. (*convert other bases to decimal*)

(a) \((11010)_2\)

(b) \((1001)_2\)

**Solution:**

\[(11010)_2 = 1 \cdot 2^4 + 1 \cdot 2^3 + 0 \cdot 2^2 + 1 \cdot 2^1 + 0 \cdot 2^0 = 16 + 8 + 2 = 26\]

\[(1001)_2 = 1 \cdot 2^3 + 0 \cdot 2^2 + 0 \cdot 2^1 + 1 \cdot 2^0 = 8 + 1 = 9\]

**Example 3:** Express the octal expansion \((314)_8\) to binary expansion and hexadecimal expansion. (*convert between bases 2, 8 and 16*)

**Solution:**

\[(314)_8 = 3 \times 8^2 + 1 \times 8 + 4.\]

Since \(3 = (11)_2\), \(1 = (1)_2\) and \(4 = (100)_2\), express them in base 2 as

\[11 \ 001 \ 100\]

So \((314)_8 = (11001100)_2\).

Regroup the binary numbers in groups of 4 digits as

\[
\begin{array}{c}
1100 \\
\hline
1001 \\
\hline
1000
\end{array}
\]

\((12)_{10} = \text{C}_{16}\)

\((12)_{10} = \text{C}_{16}\)

So \((314)_8 = (\text{CC})_{16}\).

**3 Review**

**Example 4:** Let \(a\) and \(b\) be integer not both 0. Prove that if \(\gcd(a, 4) = \gcd(b, 4) = 2\), then \(\gcd(a+b, 4) = 4\).

**Solution:**

Since \(\gcd(a, 4) = 2\), therefore \(a = 2k\) with \(k\) being odd (otherwise the \(\gcd\) will be 4). Likewise \(b = 2m\) where \(m\) is odd. Now \(a+b = 2(k+m)\). But \(k+m\) is even. So now can you deduce the result.

\[k + m = 2n\ 	ext{with } n \in \mathbb{N}, a + b = 4n.\]

\[\gcd(a + b, 4) = \gcd(4n, 4) = 4.\]

**Example 5:** Show that \(5n+3\) and \(7n+4\) are relatively prime for \(n \in \mathbb{N}\).

**Definition of relatively prime:** Two integers \(a\) and \(b\), not both 0, are relatively prime if \(\gcd(a, b) = 1\).

**Solution 1:** Using the Euclidean Algorithm:

\[(7n + 4) = 1 \cdot (5n + 3) + (2n + 1)\]

\[(5n + 3) = 2 \cdot (2n + 1) + (n + 1)\]
\[(2n + 1) = 1 \cdot (n + 1) + n \]
\[(n + 1) = 1 \cdot n + 1 \]

Therefore,
\[
gcd(7n + 4, 5n + 3) = gcd(5n + 3, 2n + 1) \]
\[
= gcd(2n + 1, n + 1) \]
\[
= gcd(n + 1, n) \]
\[
= gcd(n, 1) = 1 \]

**Corollary 7.50** \(gcd(a, b) = 1\) iff \(ax + by = 1\) for some integers \(x\) and \(y\).

**Solution 2:** Show \((5n+3)x+(7n+4)y=1\) for some integers \(x\) and \(y\).

Based on the Euclidean algorithm, we just go in the reverse order:
\[
1 = 1 = (n + 1) - n \]
\[
= (n + 1) - ((2n + 1) - (n + 1)) \]
\[
= 2(n + 1) - (2n + 1) \]
\[
= 2((5n + 3) - 2(2n + 1)) - (2n + 1) \]
\[
= 2(5n + 3) - 5(2n + 1) \]
\[
= 2(5n + 3) - 5(7n + 4) \]
\[
= 7(5n + 3) - 5(7n + 4) \]

Therefore, \(5n + 3\) and \(7n + 4\) are relatively prime. (If using this method, we can prove that \(5n + 3\) and \(7n + 4\) are relatively prime for all \(n \in \mathbb{Z}\))

**Example 6:** For functions \(f(n), g(n), h(n) : \mathbb{N} \to \mathbb{R}^+\). Show that if \(f = O(g)\) and \(g = O(h)\), then \(f = O(h)\).

**Solution:** According to the definition, \(f = O(g)\) means there exists \(C_1 > 0\) and \(k_1 \in \mathbb{N}\) such that
\[
f(n) \leq C_1 g(n) \hspace{1cm} \text{for all} \hspace{1cm} n \geq k_1.\]

Similarly, \(g = O(h)\) means there exists \(C_2 > 0\) and \(k_2 \in \mathbb{N}\) such that
\[
g(n) \leq C_2 h(n) \hspace{1cm} \text{for all} \hspace{1cm} n \geq k_2.\]

Based on the two inequalities above, we have
\[
f(n) \leq C_1 C_2 h(n) \hspace{1cm} \text{for all} \hspace{1cm} n \geq k_1 \text{ and } n \geq k_2.\]

Now if we let \(C_0 = C_1 C_2 > 0\) and \(k_0 = \max(k_1, k_2) \in \mathbb{N}\), we have
\[
f(n) \leq C_0 h(n) \hspace{1cm} \text{for all} \hspace{1cm} n \geq k_0.\]

Therefore, \(f = O(h)\).