1 Quiz 8

1. Prove for any sets $A, B$ that $A = A \cup B$ iff $B \subseteq A$.

   **Solution:** There are two directions we need to prove: (a) $A = A \cup B \Rightarrow B \subseteq A$, (b) $B \subseteq A \Rightarrow A = A \cup B$.

   (a) First, we prove $A = A \cup B \Rightarrow B \subseteq A$.

   Let $x \in B$, then by the definition of union of sets, $x \in A \cup B$. Then since $A \cup B = A$, we have $x \in A$. Therefore, $B \subseteq A$.

   (b) Then, we prove $B \subseteq A \Rightarrow A = A \cup B$.

   Let $x \in A$, then $x \in A \cup B$, so $A \subseteq A \cup B$.

   Then let $x \in A \cup B$, which is equivalent to $x \in A$ or $x \in B$. Then since $B \subseteq A$, it proves $x \in A$ or $x \in A$, which means $x \in A$. Therefore $A \cup B \subseteq A$.

   Combining the conclusions above, we have $A = A \cup B$.

   The arguments above prove $A = A \cup B \equiv B \subseteq A$.

2. Suppose that $\gcd(a, b) = 1$ and $p$ is prime. Prove that $p^2 | ab$ implies that $p^2 | a$ or $p^2 | b$.

   **Solution:** Since $\gcd(a, b) = 1$, according to the fact that $\gcd(a, b) \cdot \text{lcm}(a, b) = ab$, we know

   $$\text{lcm}(a, b) = ab.$$ 

   Then since every number can be written as the product of a series of primes, we can rewrite $a$ and $b$ as following

   $$a = \prod_i p_i^{a_i} \quad \text{and} \quad b = \prod_i p_i^{b_i},$$

   and therefore

   $$ab = \prod_i (p_i^{a_i+b_i}).$$

   Also, since $\text{lcm}(a, b) = \prod_i p_i^{\max(a_i, b_i)}$, we know

   $$a_i + b_i = \max(a_i, b_i) \text{ for } i = 1, 2, \ldots, k.$$
Now assume the power of $p$ in $a$ and $b$ are $a_k$ and $b_k$ respectively, we know

$$a_k + b_k = \max(a_k, b_k).$$

So either $a_k = 0$ or $b_k = 0$. Also, since $p^2 \mid ab$, we know that $a_k + b_k \geq 2$, and then

$$\begin{cases} a_k = 0 \\ b_k \geq 2 \end{cases} \quad \text{or} \quad \begin{cases} a_k \geq 2 \\ b_k = 0 \end{cases}.$$

Therefore, either $p^2 \mid b$ or $p^2 \mid a$. \hfill \Box

# 2 The Multiplication Principle

## 2.1 Simple Cases (two tasks)

A procedure consists of two independent tasks. Then the number of ways to perform this procedure is $n_1n_2$, where $n_1$ and $n_2$ are the number of ways to perform each task.

**Eg. 1.** A new smartphone has 3 different colors and 4 different memory sizes. How many possibilities for the combinations of the color and the memory size are there?

**Solution:** $3 \times 4 = 12.$

## 2.2 The General Multiplication Principle

For a procedure consists of $m \geq 2$ independent tasks $T_1, T_2, \ldots, T_m$, the number of ways to perform the procedure is $\prod_{i=1}^{m} n_i = n_1n_2 \ldots n_m$, where $n_i$ is the number of ways to perform $T_i$, with $i = 1, 2, \ldots, m$.

**Eg. 2.** The automobile license plate of a certain state consists of 3 numbers followed by 3 letters. How many different license plates are available under this system?

**Solution:** For the first 3 digits, each digit has 10 options. For the last 3 digits, each digit has 26 options.

\[
\begin{array}{cc}
\text{Numbers} & \text{Letters} \\
\end{array}
\]

Therefore, the total number of possibilities is

$$10 \times 10 \times 10 \times 26 \times 26 \times 26 = 10^3 \times 26^3 = 17,576,000$$

## 2.3 Theorem 8.5 and 8.7

- For the finite nonempty sets $A, B$, $|A| = m$ and $|B| = n$, the number of functions from $A$ to $B$ is $n^m$.
- For the finite nonempty sets $A, B$, $|A| = m$ and $|B| = n \ (m \leq n)$, the number of one-to-one functions from $A$ to $B$ is $\frac{n!}{(n-m)!}$. 

2
Eg. 3. Determine the number of possibilities for the following problem:

(a) 10 students take a course requiring 1 of 3 different textbooks. Each student only bought one of the textbooks, then how many different possibilities for the textbooks are there?

(b) 3 students are to attend a meeting and there are 5 seats. How many different possibilities for their seating are there?

Solution: The difference for the two cases is that different students can buy the same textbook, but they cannot take the same seat during the meeting. Therefore, the first situation is a general function and the second is a one-to-one function.

(a) The number of possibilities is $3^{10} = 59049$.

(b) The number of possibilities is $\frac{5!}{2!} = 60$.

3 The Addition Principle

3.1 Simple Cases (two tasks)

A procedure consists of two tasks and is performed if either of them is performed. The number of different ways to perform the task is $n_1 + n_2$, where $n_1$ and $n_2$ are the number of ways to perform each task.

Eg. 4. Determine the number of ways to travel from city A to city B under the following circumstances:

1. You can either drive or fly to city B from city A. There are 3 different flights available and 2 different paths for driving.

2. You determine to take plane to city B and there is no direct flight available so you need to connect at city C. There are 3 different flights from A to C and 2 different flights from C to B.

Solution: In the first situation, you only need to choose one of the traveling methods so the addition principle applies to this scenario. In the second situation, you need to choose an option for both of the cities, so the multiplication principle applies to this scenario.

(a) The number of combinations is $3 + 2 = 5$.

(b) The number of possibilities is $3 \times 2 = 6$.

3.2 The General Addition Principle

For a procedure consists of $m \geq 2$ tasks $T_1, T_2, \ldots, T_m$, and no two can be performed at the same time, the number of ways to perform the procedure is
\[ \sum_{i=1}^{m} n_i = n_1 + n_2 + \cdots + n_m, \] where \( n_i \) is the number of ways to perform \( T_i \) with \( i = 1, 2, \ldots, m \).

**Eg. 5.** How many different 10-bit strings begin with 1011 or 0110?

**Solution:** Such a sequence can have the following type:

\[
\begin{array}{cccccccccc}
1 & 0 & 1 & 1 & - & - & - & - & - & - \\
0 & 1 & 1 & 0 & - & - & - & - & - & - \\
\end{array}
\]

For the first type, there are 6 digits to be filled and each has 2 options, so the number of combinations is \( 2^6 = 64 \). The second type is similar to the first type, with 64 possible combinations. Therefore, the total number of strings is \( 64 + 64 = 128 \).

4 **The principle of Inclusion-Exclusion**

- A procedure consists of two tasks. \( n_1 \) is the number of ways to perform task 1, and \( n_2 \) is the number of ways to perform task 2. \( n_{12} \) is the number of ways to perform both tasks simultaneously. The total number of ways to perform the procedure is

\[ n_1 + n_2 - n_{12}. \]

- For two finite sets \( A \) and \( B \),

\[ |A \cup B| = |A| + |B| - |A \cap B|. \]

Particularly if two sets are disjoint,

\[ |A \cup B| = |A| + |B|. \]

**Eg. 6.** There are two seminars last week and 55 students went to the two seminars in total. We know 30 students went to the first seminar and 34 students went to the second seminar. How many students went to both seminars?

**Solution:** Assume the number of students went to both seminars is \( n \), then the number of students attended the seminars in total is \( 30 + 34 - n = 55 \).

We can solve \( n = 9 \). Therefore, 9 students went to both seminars.

- If \( A_1, A_2, \ldots, A_n \) are \( n \geq 2 \) finite sets, then

\[ | \bigcup_{1 \leq i \leq n} A_i | = \sum_{1 \leq i \leq n} |A_i| - \sum_{1 \leq i < j \leq n} |A_i \cap A_j| + (-1)^{n+1} \bigcap_{1 \leq i \leq n} A_i | \]

**Eg. 7.** Each of the four sets \( A_1, A_2, A_3 \) and \( A_4 \) contains four elements. The intersection of every \( i \) of these sets \( (2 \leq i \leq 4) \) consists of \( 5 - i \) elements. What is \( |A_1 \cup A_2 \cup A_3 \cup A_4| \)?
Solution: Based on the formula above,
\[ |A_1 \cup A_2 \cup A_3 \cup A_4| = \sum_{1 \leq i \leq 4} |A_i| - \sum_{1 \leq i < j \leq 4} |A_i \cap A_j| + \sum_{1 \leq i < j < k \leq 4} |A_i \cap A_j \cap A_k| - |A_1 \cap A_2 \cap A_3 \cap A_4| \]
\[ = 4 \times 4 - 6 \times (5 - 2) + 4 \times (5 - 3) - (5 - 4) \]
\[ = 5 \]

5 The pigeonhole Principle

- If a set \( S \) with \( n \) elements is divided into \( k \) pairwise disjoint subsets \( S_1, S_2, \ldots, S_k \), then at least one of them has at least \( \lceil n/k \rceil \)

- A set \( S \) with \( n \) elements is partitioned into \( k \) pairwise disjoint subsets \( S_1, S_2, \ldots, S_k \), where \( |S_i| \geq n_i \) for a positive integer \( n_i \) for \( i = 1, 2, \ldots, k \). Then each subset of \( S \) with at least

\[ 1 + \sum_{i=1}^{k} (n_i - 1) \]

elements contains at least \( n_i \) elements of \( S_i \) for some integer \( i \) with \( 1 \leq i \leq k \).

Eg. 8. How many people must be present to guarantee that
(a) at least two have the same birthday?
(b) at least two of their birthdays are in the same month?
(c) at least three of their birthdays are in one of the months January, February, March, April or at least four of their birthdays are in one of the remaining months?

Solution:
(a) There are 366 days in a year considering the leap years. Assume there are \( n \) people, then according to the Pigeonhole Principle, at least \( \lceil n/366 \rceil \) have the same birthday.
\[ \lceil n/366 \rceil = 2, \]
so \( n > 366 \) and there must be at least 366 people.

(b) Since there are 12 months, similar to the previous question, there must be at least 13 people.

(c) According to the Pigeonhole Principle, the number of people should be at least
\[ (3 - 1) \times 4 + (4 - 1) \times 8 + 1 = 33. \]