1 Quiz 1

1. Show that \( \sim (P \land (\sim Q)) \) is not logically equivalent to \( (\sim P) \land Q \).

**Solution:** When you need to prove two statements are not logically equivalent, you only need to give a counter example. So you can try listing the truth values in the truth table. You’ll find that the first row is already a counter example. *Q.E.D*

<table>
<thead>
<tr>
<th>( P )</th>
<th>( Q )</th>
<th>( \sim P )</th>
<th>( \sim (P \land (\sim Q)) )</th>
<th>( (\sim P) \land Q )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( T )</td>
<td>( T )</td>
<td>( F )</td>
<td>( F )</td>
<td>( T )</td>
</tr>
<tr>
<td>( F )</td>
<td>( T )</td>
<td>( T )</td>
<td>( T )</td>
<td>( F )</td>
</tr>
</tbody>
</table>

2. Show that \( P \Rightarrow Q \) is not logically equivalent to \( Q \Rightarrow P \).

**Solution:** Similar to the previous problem, we only need to find a counter example.

<table>
<thead>
<tr>
<th>( P )</th>
<th>( Q )</th>
<th>( P \Rightarrow Q )</th>
<th>( Q \Rightarrow P )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( T )</td>
<td>( T )</td>
<td>( T )</td>
<td>( T )</td>
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<tr>
<td>( T )</td>
<td>( F )</td>
<td>( F )</td>
<td>( T )</td>
</tr>
</tbody>
</table>

The second row is a counter example. *Q.E.D*

2 Power Sets

- \( \mathcal{P}(A) = \{ B : B \subseteq A \} \) (\( \mathcal{P}(A) \) is the power set of \( A \)).
- \( A \in \mathcal{P}(A) \) and \( \emptyset \in \mathcal{P}(A) \).
- If \( |A| = n \) then \( |\mathcal{P}(A)| = 2^n \).

**Eg. 1.** Determine the power set of \( A = \{ n \in \mathbb{Z} : |n - 1| < 2 \} \).

**Solution:** As \( |n - 1| < 2 \equiv -1 < n < 3 \) and \( n \in \mathbb{Z} \), \( A = \{0, 1, 2\} \). Thus, we have

\[
\mathcal{P}(A) = \{ \emptyset, \{0\}, \{1\}, \{2\}, \{0, 1\}, \{0, 2\}, \{1, 2\}, \{0, 1, 2\} \} 
\]
3 Set Operations

- Intersection $A \cap B = \{x : x \in A \text{ and } x \in B\}$.
- Union $A \cup B = \{x : x \in A \text{ or } x \in B\}$.
- $(A \cap B) \subseteq (A \cup B)$.
- Commutative Laws:
  \[ A \cap B = B \cap A \text{ and } A \cup B = B \cup A. \]
- Associative Laws:
  \[ (A \cap B) \cap C = A \cap (B \cap C) \]
  \[ (A \cup B) \cup C = A \cup (B \cup C) \]
- Distributive Laws:
  \[ A \cap (B \cup C) = (A \cap B) \cup (A \cap C) \]
  \[ A \cup (B \cap C) = (A \cup B) \cap (A \cup C) \]
- $A$ and $B$ are disjoint if $A \cap B = \emptyset$.
- The difference $A - B = \{x : x \in A \text{ and } x \notin B\}$.
- The symmetric difference $A \oplus B = (A - B) \cup (B - A)$.

**Eg. 2.** For $A = \{1, 2\}, B = \{-1, 0, 1\}$ and the universal set $U = \{-2, -1, 0, 1, 2\}$, determine

\[
(a) A \cup B \quad (b) A \cap B \quad (c) A - B \quad (d) B \quad (e) A \oplus B
\]

**Solution:**

(a) $A \cup B = \{-1, 0, 1, 2\}$.
(b) $A \cap B = \{1\}$.
(c) $A - B = \{2\}$.
(d) $B = \{-2, 2\}$.
(e) $A \oplus B = \{-1, 0, 2\}$.
4 Complement of a Set

Definition: \( \overline{A} = \{ x \in U : x \notin A \} = U \setminus A \)

De Morgan’s Laws:

\[ \overline{A \cup B} = \overline{A} \cap \overline{B} \quad \text{and} \quad \overline{A \cap B} = \overline{A} \cup \overline{B} \]

Eg. 3. For \( n \in \mathbb{N} \), let \( A_n = \left\{ \frac{2}{n+2} \right\} \) and \( B_n = \left\{ \frac{1}{n} \right\} \). Determine \( \left( \bigcup_{n=1}^{3} A_n \right) \cap \left( \bigcup_{n=1}^{3} B_n \right) \).

Solution:
We have \( A_1 = \left\{ \frac{2}{3} \right\}, \ A_2 = \left\{ \frac{1}{2} \right\}, \ A_3 = \left\{ \frac{2}{5} \right\}, \) so

\[ \bigcup_{n=1}^{3} A_n = \left\{ \frac{2}{3} \cdot \frac{1}{2} \cdot \frac{2}{5} \right\}. \]

\( B_1 = \{1\}, \ B_2 = \{ \frac{1}{2} \}, \ B_3 = \{ \frac{1}{3} \}, \) so

\[ \bigcup_{n=1}^{3} B_n = \left\{ 1, \frac{1}{2}, \frac{1}{3} \right\}. \]

Therefore the solution is \( \left( \bigcup_{n=1}^{3} A_n \right) \cap \left( \bigcup_{n=1}^{3} B_n \right) = \left\{ \frac{1}{2} \right\} \).

Eg. 4.

(a) Illustrate the following by drawing Venn diagrams. For two sets \( A \) and \( B \), \( A \setminus B = A \cap \overline{B} \).

(b) For sets \( A \) and \( B \), show that \( A \setminus B = A \cap \overline{B} \).

Solution:

(b) For an element \( x \in (A \setminus B) \), we have

\[ x \in (A \setminus B) \]
\[ \equiv x \in A \text{ and } x \notin B \]
\[ \equiv x \in A \text{ and } x \in \overline{B} \]
\[ \equiv x \in A \cap \overline{B} \]

5 Cartesian Products

- For an ordered pair \((a, b)\), if \( a \neq b \), then \((a, b) \neq (b, a)\).
- Cartesian product: \( A \times B = \{(a, b) : a \in A \text{ and } b \in B\} \).
- \(|A \times B| = |A| \cdot |B| = |B| \cdot |A| = |B \times A|\).

Eg. 5.
(a) \( A = \{1,2\} \) and \( B = \{1\} \), determine \( P(A \times B) \).

(b) For \( A = \emptyset \) and \( B = \{0\} \), determine \((A \times B) \cap P(A \times B) \).

**Solution:**

(a) \( A \times B = \{(1,1), (2,1)\} \). Then \( P(A \times B) = \{\emptyset, \{(1,1)\}, \{(2,1)\}, \{(1,1), (2,1)\}\} \).

(b) \( A \times B = \emptyset \), so \( P(A \times B) = \emptyset \). Thus \((A \times B) \cap P(A \times B) = \emptyset \).

**Eg. 6.** Show that for sets \( A, B \) and \( C \), \( (A \times B) \cap (A \times C) = A \times (B \cap C) \).

**Solution:**

For an ordered pair \((a, b)\), we have:

\[
(x, y) \in (A \times B) \equiv x \in A \text{ and } y \in B \\
(x, y) \in (A \times C) \equiv x \in A \text{ and } y \in C.
\]

So

\[
(x, y) \in (A \times B) \cap (A \times C) \\
\equiv (x \in A \text{ and } y \in B) \text{ and } (x \in A \text{ and } y \in C) \\
\equiv x \in A \text{ and } y \in B \text{ and } y \in C \\
\equiv x \in A \text{ and } y \in (B \cap C) \\
\equiv (x, y) \in A \times (B \cap C).
\]

Therefore, \((A \times B) \cap (A \times C) = A \times (B \cap C) \).

6 **Partitions**

**Definition:** A partition of a nonempty set \( A \) is a collection of nonempty subsets of \( A \) such that every element of \( A \) belongs to exactly one of these subsets.

For a partition \( \mathcal{P} = \{S_1, S_2, \ldots, S_k\} \) \( (k \geq 1) \) of a nonempty set \( A \),

- Every \( S_i \) is nonempty.
- Every two different subsets \( S_i \) and \( S_j \) are disjoint.
- \( \bigcup_{i=1}^{k} S_i = A \).

**Eg. 7.** For the set \( A = \{1,2,\ldots,10\} \), determine which of the following are partitions of \( A \).

(a) \( \mathcal{P}_1 = \{\{1,2,3,7\}, \{5,6\}, \{9,10\}\} \). (No. 4 and 8 are not included.)

(b) \( \mathcal{P}_2 = \{\{1,2,3,4\}, \{4,5,6,7\}, \{8,9,10\}\} \). No. 4 appears in two subsets.

(c) \( \mathcal{P}_3 = \{\{1,2,3,4\}, \{5,6,9\}, \emptyset, \{7,8,10\}\} \). No. An empty set is included.

(d) \( \mathcal{P}_4 = \{\{1,2,3,4\}, \{5,6,9\}, \{7,8,10\}\} \). Yes.