1. We call exchange matrix $E$ the $n \times n$ matrix with entries $\eta_{ij} = 1$ if $i + j - 1 = n$ and $\eta_{ij} = 0$ otherwise. So $E$ looks like a mirror image of the $n \times n$ identity matrix in that it has its ones on the anti-diagonal. An $n \times n$ matrix is said to be persymmetric if $EA \equiv A^T$.

(a) What are the rows of $EA$? What are the columns of $AE$? (Hint: Answer these in plain words no justifications needed).

(b) What is the inverse of $E$? [Hint: exploit (a)]

(c) Show that $A$ is persymmetric if $AE$ is symmetric. Conclude from this that a matrix is persymmetric if it is symmetric about its anti-diagonal

(d) Are Toeplitz matrices persymmetric? Find a persymmetric matrix that is not Toeplitz.

(e) Prove that the inverse of a Toeplitz matrix is persymmetric.

(f) Prove that the inverse of a Toeplitz matrix is persymmetric. Is it Toeplitz in general (If your answer is yes, prove result, else provide counter example)?

2. Let $A$ be a real unitary matrix. Show that $\det(A) = \pm 1$. Let $B$ be another unitary real matrix and assume that $\det(B) = -\det(A)$. Then show that $A + B$ is singular. [Hint: show that $\det(A + B) = 0$ by using properties of determinants (products...)]

3. Given two vectors $u, v, \in \mathbb{R}^n$, (a) What is $\det(uv^T)$? (b) What are all the eigenvalues of $uv^T$? (c) What is Trace($uv^T$)? d) what is $\det(I + uv^T)$? e) What is Trace($I + uv^T$)? (f) What is the determinant of $A + uv^T$ when $A$ is nonsingular? [Hint: for (d) and (f), exploit the fact that $\det(A) = \text{product of all eigenvalues.}]

4. What are all real normal matrices of dimension 2?

5. Let X be an $m \times n$ matrix, with $m \geq n$, that is of full rank. Show that $X^TX$ is nonsingular. [Hint: By making judicious use of inner products, show that $X^TXy = 0$ implies that $Xy = 0$ which in turn implies that $y = 0$.]

6. Let a full-rank matrix $X \in \mathbb{R}^{m \times p}$ and a full-rank matrix $Y \in \mathbb{R}^{n \times p}$, with $m, n \geq p$, and let $A = XY^T$. Show that the rank of $A$ is $p$. [Hint: The rank is clearly $\leq p$ because the range of $A$ is included in the span of $X$. To show that it is exactly $p$, you will need to show that each of the columns of $X$ is ‘reached’, i.e., that there is a vector $z_j$ such $XY^Tz_j = x_j$ where $x_j = j$-th column of $X$. You will need to use the fact that $Y^TY$ is nonsingular - a consequence of the previous question.

7. (Continuation of previous exercise) Show that a matrix $A \in \mathbb{R}^{m \times n}$ is of rank $p$ if and only if there are two full rank matrices $X$ and $Y$ where $X \in \mathbb{R}^{m \times p}$ and $Y \in \mathbb{R}^{n \times p}$ such that $A = XY^T$. [Hint: The if part was shown in previous exercise.]

8. Let $A$ be an $n \times n$ matrix whose only nonzero entries are in the first column and first row (i.e., $a_{ii} = 0$ when $i > 1$ and $j > 1$). (a) Show that $A$ is of rank $\leq 2$. When is the rank less than 2? (b) Assume that in addition $A$ is symmetric and that $a_{11} = 1$. Show that there exist two vectors $u$ and $v$ such that $A = uu^T - vv^T$. 
9. Let $T$ be a symmetric Toeplitz matrix $T = [t_{i-j}]_{i,j=1}^n$ with $t_0 = 1$ and define the $n \times n$ lower triangular shift matrix $Z$:

$$
Z = \begin{pmatrix}
0 & 0 & 0 & \cdots & 0 \\
1 & 0 & 0 & \cdots & 0 \\
0 & 1 & 0 & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots \\
0 & \cdots & \cdots & 0 & 1
\end{pmatrix}
$$

(a) Show that the matrix $D = T - ZTZ^T$ has rank $\leq 2$. Hints: For a matrix $X$ what are the columns of $XZ^T$? What are the rows of $ZX$? Then use the result of the previous exercise.

(b) Show that $D$ can be written as $D = uu^T - vv^T$ for certain vectors $u$, $v$ to be specified. (Note: $D$ is called the displacement of $T$ with respect to $Z$ and the rank of $D$ is the displacement rank of $T$. Matrices with low displacement ranks have been extensively studied.)

10. Are the following functions from $\mathbb{R}^n$ to $\mathbb{R}$ vector norms? (prove or disprove).

   (a) $N(x) = \sum_{i=1}^n \left| \frac{x_i}{2} \right|$;  
   (b) $N(x) = \left( \sum_{i=1}^n |x_i|^{1/2} \right)^2$;  
   (c) $N(x) = \left( \sum_{i=1}^n |x_i| \right)^2$.

11. For the following exercise, you need to use Matlab. Consider the matrix:

$$
\begin{pmatrix}
-1 & 1 & -4 \\
0 & 0 & 0 \\
-2 & 4 & -4 \\
1 & -2 & 2 \\
-1 & 2 & -2
\end{pmatrix}
$$

   a. Use Matlab to compute the 1-norm, the 2-norm, the infinity norm and the Frobenius norm of the above matrix.

   b. Find the eigenvalues and the spectral radius of $A(1 : 3, 1 : 3)$ [matlab notation].

   c. Find the determinant of $A^T A$. Without using matlab, find the determinant of $AA^T$. Explain and verify your result with matlab. What can you state about the rank of $A$?

   d. Use the ‘rank’ function to determine the rank of $A$.

   e. Explore the “reduced row echelon form” function of matlab called rref. Once you understand what the rref function does, use it to find the RREF form of $A$. Can you explain in words what was done to obtain this form? [recall Gauss-Jordan elimination].

   For questions like these, you use Matlab as a sort of calculator. Show what command you used and the result. Also write down answers to other questions asked.