Note: Please review what you know about the LU factorization before starting this HW. We will review the LU factorization in the class of Monday Oct. 1.

1. Let \( \| \cdot \| \) be a norm in \( \mathbb{R}^n \). Define:
   \[
   \| x \|' = \sup \{ u^T x : u \in \mathbb{R}^n, \| u \| = 1 \}
   \]
   (a) Prove that this equation defines a norm (called the “dual norm” of \( \| \cdot \| \)).
   (b) Show that for all \( x, y \in \mathbb{R}^n \) we have
   \[|x^T y| \leq \| x \| \| y \|'
   \]
   (c) What is the dual norm of \( \| \cdot \|_1 \)?

2. (a) Find the LU factorization of the matrix:
   \[
   A = \begin{pmatrix}
   4 & -4 & 0 & 2 \\
   -2 & 6 & 2 & -3 \\
   2 & 0 & 5 & 8 \\
   0 & 2 & -1 & -3
   \end{pmatrix}
   \]
   (b) What is the determinant of \( A \)?
   (c) Solve the linear system \( Ax = b \) where \( b = [6, -7, 9, -4]^T \) using the LU factors obtained in part (a) above.
   (d) Using the LU factors obtained in (a) find the last column of the inverse of \( A \), without computing the whole inverse.

3. Write a matlab function which computes the LU factorization (without pivoting) of a matrix. Test it on the matrix of the previous question and verify the answers to subquestions (a) and (d). [Hint: Your starting point should be the gauss.m script that is posted. Your script should take a matrix \( A \) and return the matrices \( L \) and \( U \) – so for example \([L, U] = \text{gaussLU}(A)\)]

4. (a) Determine the LU factorization (Gaussian elimination without pivoting) of the following matrix
   \[
   A = \begin{pmatrix}
   1 & -1 & 1 \\
   0 & 4 & 2 \\
   6 & 2 & 0
   \end{pmatrix}
   \]
   (b) Compute the determinant of \( A \)
   (c) Compute the inverse of \( A \).
   (d) Repeat the above questions when partial pivoting is used, i.e., find the permutation matrix \( P \) and the matrices \( L, U \) such that \( PA = LU \), compute the determinant of \( A \) based on this factorization, and compute the inverse of \( A \), based on this factorization.
   (e) Use the answer from the previous question (result of LU factorization *with* partial pivoting) to solve the system \( Ax = b \) when \( b = [-2, 2, 2]^T \).
5. Let $A = LU$ the factorization of an $n \times n$ matrix $A$ with $|l_{ij}| \leq 1$. Let $a_i^T$ and $u_i^T$ denote the $i$-th rows of $A$ and $U$ respectively. Verify that the following equation is satisfied:

$$u_i^T = a_i^T - \sum_{j=1}^{i-1} l_{ij} u_j^T.$$ 

Use this result to show that $\|U\|_\infty \leq 2^{n-1} \|A\|_\infty$. [Hint: You can use induction]

6. (a) Apply the matlab script you developed in Question 3 to compute the LU factorization of the following matrix (for the case when $n = 8$):

$$A = \begin{pmatrix}
1 & 0 & \cdots & \cdots & 0 & 1 \\
-1 & 1 & 0 & \cdots & 0 & 1 \\
-1 & -1 & 1 & 0 & \cdots & \vdots \\
\vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\
-1 & \cdots & \cdots & -1 & 1 \\
\end{pmatrix}$$

(b) What can you say about $\|U\|_\infty$ as a function of the size $n$? How do you relate this result to that of Question 5?

(c) Compute, for the case $n = 8$, the matrix $128 \ast A^{-1}$ (in matlab). Can you tell what the inverse of $A$ is in general (for any $n$)? Prove your result. Ignore the following subquestions: What is (exactly) the 1-norm condition number of $A$ as a function of $n$. Any comments?

7. Note: ignore this question. It is postponed to HW3. Consider the following matlab function which returns the absolute value of its first argument:

```matlab
function z = absolute(x,m)
    y = x .^ 2;
    for i=1:m
        y = sqrt(y);
    end
    z = y;
    for i=1:m-1
        z = z .^ 2;
    end
end
```

Apply this function for the vector $x = [.25, .50, .75, 1.25, 1.50, 1.75, 2]$ and for $m = 50$. Give an error analysis to explain the result. [Hint: This will be discussed a little in class]