1. Consider a linear system $Ax = b$ where $A$ is *tridiagonal*.
   (a) Carry out an exact operation count for solving the linear system by Gaussian elimination (Note: Operations with zeros are not performed).
   (b) What are the patterns of the matrices $L$ anf $U$ of of the (standard) LU factorization of $A$?
   (c) Answer the same question as (a) when partial pivoting is used. [Note: for all practical purposes, you can assume that you swap rows every time when you compute the cost]
   (d) Assume now that you use Gauss-Jordan elimination (no pivoting) - Show the pattern of the matrix you obtain at the step $k = 3$ for a case when $n = 6$. [Hint: indicate a nonzero entry by a * or an x] What is the operation count for solving the system by the Gauss-Jordan method? [notes: 1) $c = 0 - a * b = -a * b$ counts as one operation; 2) Ignore the final stage of solving with the final diagonal matrix.]

2. An $n \times n$ matrix $A$ is said to be ‘strictly row diagonally dominant’, if:
   $$|a_{ii}| - \sum_{j=1, j\neq i}^{n} |a_{ij}| \equiv \delta_i > 0, \quad \text{for } i = 1, \ldots, n.$$ 
   Consider such a matrix and define $\delta_{\min} = \min_{i=1, \ldots, n} \delta_i$.
   (a) Show that for any vector $y$ we have $\|Ay\|_\infty \geq \delta_{\min} \|y\|_\infty$
   (b) Deduce from (a) that $A$ is nonsingular.
   (c) Show that $\|A^{-1}\|_\infty \leq \frac{1}{\delta_{\min}}$.
   (d) [5 Bonus pts] Show that when Gaussian elimination with partial pivoting is applied to a strictly column diagonally dominant ($A^T$ is row-diagonal dominant) then rows are never permuted. [Hint: Show (by induction) that the matrices $A_k$ obtained at each step of GE remain column Diag. Dom.]

3. Assume the IEEE standard for floating point arithmetic see pages 4-32 and 4-31 of Lecture notes set # 4. If you use the matlab command `num2hex` you will find that the IEEE hex representation of the number `-6.0` (negative 6) is
   `'-c018000000000000'`
   The exponent+sign part is `c01` and the mantissa is 8 followed by 12 zeros (in HEX). Explain why you obtain this representation. [Hint: In binary the very first bit is one which indicates a negative sign]. What is the internal representation of -6 in single precision?

4. In this exercise we assume the IEEE standard for floating point arithmetic in single precision, see page 4-31 of Lecture notes set # 4. With respect to the earlier representation $\beta = 2$, and $t = 24$ (one bit is ‘hidden’).
   (a) What is the largest valid floating point number represented by this system? [Hint: ‘valid’ means the exponent is between -126 and 127 as indicated in lecture notes]
   (b) What is the smallest valid positive (nonzero) floating point number?
   (c) Not all positive integers are represented exactly in this floating point system. What is the smallest (positive) integer that is not exactly represented?
5. Consider the following matlab function which returns the absolute value of its first argument:

```matlab
function z = absolute(x,m)
    y = x .^ 2;
    for i=1:m
        y = sqrt(y);
    end
    z = y;
    for i=1:m-1
        z = z .^ 2;
    end
end
```

Apply this function for the vector $x = [.25, .50, .75, 1.25, 1.50, 1.75, 2]$ and for $m = 50$. Give an error analysis to explain the result. [Hint: This will be discussed a little in class]

6. Let $A$ and $B$ two nonsingular matrices of size $n \times n$. Show that

$$fl(AB) = (A + E_A)B \quad \text{where} \quad |E_A| \leq \gamma_n |A| |B| |B^{-1}|$$

($\gamma_n$ defined in the notes) and derive a corresponding bound in which $B$ is perturbed. [This result shows the limitation of backward error analysis. In this case it is clear that forward error analysis yields a ‘cleaner’ result].

7. Consider the linear system $Ax = b$ where $|\varepsilon| < 4$ and:

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 4 - \varepsilon \end{pmatrix} \quad b = \begin{pmatrix} 3 \\ 6 - \varepsilon \end{pmatrix}$$

(a) Solve the above linear system by Gaussian elimination with partial pivoting.

(b) Now replace the $6 - \varepsilon$ in the right-hand side by 6 and solve the new system. How does the result depend on $\varepsilon$?

(c) Verify (a) and (b) with the system you obtain when $\varepsilon = 0.001$ [you can use matlab for this if desired].

(d) What is the 1-norm of the inverse of the matrix you obtain in (c)? Can you explain what you observe based on your answer?