1. Find a $2 \times 2$ real matrix which satisfies $(Ax, x) > 0$ for all real nonzero vectors $x$ - but which is not positive definite when regarded as a member of $\mathbb{C}^{2 \times 2}$. [Hint: note that for $A$ to be positive definite, $(Ax, x)$ must be first real and then also positive.]

2. Let $A \in \mathbb{R}^{n \times n}$ be a symmetric positive definite matrix.
   (a) Show that $|a_{ij}| < \sqrt{a_{ii} a_{jj}}$
   (b) Show also that $|a_{ij}| < (a_{ii} + a_{jj})/2$

3. Show that the following symmetric matrices are not positive definite [without computing eigenvalues]
   \[ A = \begin{pmatrix} 0 & 3 & 1 \\ 3 & 2 & 0 \\ 1 & 0 & 2 \end{pmatrix} \quad B = \begin{pmatrix} 2 & 3 & 1 \\ 3 & 2 & 0 \\ 1 & 0 & 2 \end{pmatrix} \quad C = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 2 \end{pmatrix} \]

4. This exercise is an introduction to the notion of numerical range otherwise known as field of values. The numerical range of a (square) matrix $A \in \mathbb{C}^{n \times n}$ is the set
   \[ W(A) = \{ z^H A z \mid z \in \mathbb{C}^n, \|z\|_2 = 1 \} \]
   (a) Show that $W(A)$ is a bounded set of $\mathbb{C}$ (Hint: It is included in a disk. Use matrix norms).
   (b) Show that the spectrum of $A$ is included in $W(A)$.
   (c) Show that if $0 \notin W(A)$ then $A$ is nonsingular.
   (d) Show also that if $0 \notin W(A)$ then $A$ has an LU factorization.
   (e) It is known (Hausdorff’s theorem) that the numerical range is a convex set in $\mathbb{C}$. What can you say about $W(A)$ when $A$ is real symmetric positive definite?

5. The Hald cement data is used in several books and papers as an example of regression and least-squares. The right-hand side is the heat evolved in cement during hardening. The variables are coefficients $\xi_1, \ldots, \xi_4$ of four different ingredients of the mix. The right-hand side $b$ and the matrix $A \in \mathbb{R}^{13 \times 4}$ corresponding to 13 measurements, are available from the script $\texttt{Hald.m}$ which you will find in the matlab section of the class web-site. [Incidentally this is available if you have access to the Statistics and Machine Learning Toolbox – simply type $\texttt{load hald}$].
   To the 4 variables $\xi_i, i = 1 : 4$ we add a constant (call it $\xi_0$) to the model - so in effect we want to find $\xi_0, \xi_1, \ldots, \xi_4$ so that $\xi_0 + \xi_1 a_{i,1} + \cdots + \xi_4 a_{i,4} \approx b_i$ for $i = 1 : 13$. In what follows, $e$ is the vector of all ones.
   (a) Solve the least squares problem to get the $\xi_i$’s by the method of normal equations. What is $\kappa_2(A)$?
   (b) We now show how to get rid of the constant unknown from the system. Write $x = \begin{pmatrix} \xi_0 \\ y \end{pmatrix}$ where $\xi_0$ is a scalar, and show how to eliminate $\xi_0$ from the system [Hint: Start with the orthogonality conditions for optimality but restrict these to only one condition involving the vector $e$]. The resulting problem is now a least-squares problem of the form $\min \|By - c\|_2$ involving only the $y$ vector. Show the matrix $B$ and new right-hand side $c$. What is the condition number of $B$?
   (c) Continued from (b). How can you interpret $c$ relative to $b$ and $B$ relative to $A$? [Hint: What is $e^T B$? What is $e^T c$?]
6. The purpose of this exercise is to test 3 different ways of computing the QR factorization of a matrix $A$

   (a) The classical Gram-Schmidt algorithm
   (b) The modified Gram-Schmidt algorithm
   (c) The Cholesky factorization of $A^T A$

   Explain how the Cholesky factorization of $A^T A$ can be used. In the following you should use the script `cholR` that is posted (not the `chol` function from matlab). You can use `inv` to invert triangular matrices.

   A data set is posted on the class web-site (see the matlab page). Write a script which loads the matrix and then for each of the three methods above compute the $Q$ and $R$ factors and the error measures

   $$\|A - Q \ast R\|_2, \quad \|I - Q^T \ast Q\|_2$$

   Present your result in the form of a table and comment on them.

7. If $w \in \mathbb{R}^n$ and $\|w\|_2 = 1$, then the matrix $P = I - 2ww^T$, called a Householder reflector, is at the same time symmetric and unitary. [seen or to be seen in class]. Let $x \in \mathbb{R}^n$, with $x \neq 0$, and let $y = Px$. (a) When do we have $y = x$? (b) Assume now that $w^T x > 0$. Show that $y \neq x$ and that $w = (x - y)/\|x - y\|_2$. (c) Show a geometric illustration of the result in (b).