**Inner products and Norms**

**Inner product of 2 vectors**

- Inner product of 2 vectors $x$ and $y$ in $\mathbb{R}^n$:
  \[ x_1 y_1 + x_2 y_2 + \cdots + x_n y_n \]

Notation: $(x, y)$ or $y^T x$

- For complex vectors $(x, y) = x_1 \bar{y}_1 + x_2 \bar{y}_2 + \cdots + x_n \bar{y}_n$ in $\mathbb{C}^n$

Note: $(x, y) = y^H x$

**Properties of Inner Product:**

- $(x, y) = (y, x)$
- $(\alpha x, y) = \alpha \cdot (x, y)$
- $(x, x) \geq 0$ is always real and non-negative.
- $(x, x) = 0$ iff $x = 0$ (for finite dimensional spaces).
- Given $A \in \mathbb{C}^{m \times n}$ then $(Ax, y) = (x, A^H y) \forall x \in \mathbb{C}^n, \forall y \in \mathbb{C}^m$

**Vector norms**

Norms are needed to measure lengths of vectors and closeness of two vectors. Examples of use: Estimate convergence rate of an iterative method; Estimate the error of an approximation to a given solution; ...

- A vector norm on a vector space $X$ is a real-valued function on $X$, which satisfies the following three conditions:
  1. $\|x\| \geq 0$, $\forall x \in X$, and $\|x\| = 0$ iff $x = 0$.
  2. $\|\alpha x\| = |\alpha| \|x\|$, $\forall x \in X$, $\forall \alpha \in \mathbb{C}$.
  3. $\|x + y\| \leq \|x\| + \|y\|$, $\forall x, y \in X$.

- Third property is called the triangle inequality.

**Important example: Euclidean norm** on $X = \mathbb{C}^n$,

\[ \|x\|_2 = (x, x)^{1/2} = \sqrt{|x_1|^2 + |x_2|^2 + \cdots + |x_n|^2} \]

Show that when $Q$ is orthogonal then $\|Qx\|_2 = \|x\|_2$

- Most common vector norms in numerical linear algebra: special cases of the Hölder norms (for $p \geq 1$):

\[ \|x\|_p = \left( \sum_{i=1}^{n} |x_i|^p \right)^{1/p} \]

Find out (bbl search) how to show that these are indeed norms for any $p \geq 1$ (Not easy for 3rd requirement!)
**Property:**

- Limit of $\|x\|_p$ when $p \to \infty$ exists:

  $$\lim_{p \to \infty} \|x\|_p = \max_{i=1}^n |x_i|$$

- Defines a norm denoted by $\|\cdot\|_\infty$.

- The cases $p = 1$, $p = 2$, and $p = \infty$ lead to the most important norms $\|\cdot\|_p$ in practice. These are:

  $$\|x\|_1 = |x_1| + |x_2| + \cdots + |x_n|,$$
  $$\|x\|_2 = (|x_1|^2 + |x_2|^2 + \cdots + |x_n|^2)^{1/2},$$
  $$\|x\|_\infty = \max_{i=1,\ldots,n} |x_i|.$$ 

**The Cauchy-Schwarz inequality (important) is:**

$$|\langle x, y \rangle| \leq \|x\|_2 \|y\|_2.$$ 

- When do you have equality in the above relation?

- Expand $(x + y, x + y)$. What does the Cauchy-Schwarz inequality imply?

**The Hölder inequality (less important for $p \neq 2$) is:**

$$|\langle x, y \rangle| \leq \|x\|_p \|y\|_q,$$

with $\frac{1}{p} + \frac{1}{q} = 1$.

- Second triangle inequality:

  $$\|x - y\| \leq \|x\| + \|y\|.$$ 

- Consider the metric $d(x, y) = \max_i |x_i - y_i|$. Show that any norm in $\mathbb{R}^n$ is a continuous function with respect to this metric.

**Equivalence of norms:**

In finite dimensional spaces ($\mathbb{R}^n$, $\mathbb{C}^n$, ..) all norms are ‘equivalent’: if $\phi_1$ and $\phi_2$ are two norms then there exists positive constants $\alpha, \beta$ such that,

$$\beta \phi_2(x) \leq \phi_1(x) \leq \alpha \phi_2(x).$$

- How can you prove this result? [Hint: Show for $\phi_2 = \|\cdot\|_\infty$]

- We can bound one norm in terms of any other norm.

- Show that for any $x$: 

  $$\frac{1}{\sqrt{n}} \|x\|_1 \leq \|x\|_2 \leq \|x\|_1.$$ 

- What are the “unit balls” $B_p = \{x \mid \|x\|_p \leq 1\}$ associated with the norms $\|\cdot\|_p$ for $p = 1, 2, \infty$, in $\mathbb{R}^2$?

**Convergence of vector sequences**

A sequence of vectors $x^{(k)}$, $k = 1, \ldots, \infty$ converges to a vector $x$ with respect to the norm $\|\cdot\|$ if, by definition,

$$\lim_{k \to \infty} \|x^{(k)} - x\| = 0.$$ 

- Important point: because all norms in $\mathbb{R}^n$ are equivalent, the convergence of $x^{(k)}$ w.r.t. a given norm implies convergence w.r.t. any other norm.

- Notation:

  $$\lim_{k \to \infty} x^{(k)} = x$$
Example: The sequence

\[ x^{(k)} = \left( \frac{1 + 1/k}{k + \log_2 k} \right) \]

converges to

\[ x = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \]

Note: Convergence of \( x^{(k)} \) to \( x \) is the same as the convergence of each individual component \( x_i^{(k)} \) of \( x^{(k)} \) to the corresponding component \( x_i \) of \( x \).

Matrix norms

- Can define matrix norms by considering \( m \times n \) matrices as vectors in \( \mathbb{R}^{mn} \). These norms satisfy the usual properties of vector norms, i.e.,

  1. \( \|A\| \geq 0, \forall A \in \mathbb{C}^{m \times n} \), and \( \|A\| = 0 \) iff \( A = 0 \)
  2. \( \|\alpha A\| = |\alpha|\|A\|, \forall A \in \mathbb{C}^{m \times n}, \forall \alpha \in \mathbb{C} \)
  3. \( \|A + B\| \leq \|A\| + \|B\|, \forall A, B \in \mathbb{C}^{m \times n}. \)

- However, these will lack (in general) the right properties for composition of operators (product of matrices).
- The case of \( \|\cdot\|_2 \) yields the Frobenius norm of matrices.

Consistency / sub-multiplicativity of matrix norms

- A fundamental property of matrix norms is consistency

\[ \|AB\|_p \leq \|A\|_p \|B\|_p. \]

[Also termed “sub-multiplicativity”]

- Consequence: \( \|A^k\|_p \leq \|A\|_p^k \)
- \( A^k \) converges to zero if any of its \( p \)-norms is \(< 1 \)
[Note: sufficient but not necessary condition]
**Frobenius norms of matrices**

- The Frobenius norm of a matrix is defined by
  \[ \|A\|_F = \left( \sum_{j=1}^{n} \sum_{i=1}^{m} |a_{ij}|^2 \right)^{1/2}. \]
- Same as the 2-norm of the column vector in \( \mathbb{C}^{mn} \) consisting of all the columns (respectively rows) of \( A \).
- This norm is also consistent [but not induced from a vector norm]

**Expressions of standard matrix norms**

- Recall the notation: (for square \( n \times n \) matrices)
  \[ \rho(A) = \max |\lambda_i(A)|; \quad Tr(A) = \sum_{i=1}^{n} a_{ii} = \sum_{i=1}^{n} \lambda_i(A) \]
  where \( \lambda_i(A), i = 1, 2, \ldots, n \) are all eigenvalues of \( A \)

\[ \|A\|_1 = \max_{j=1,\ldots,n} \sum_{i=1}^{m} |a_{ij}|, \]
\[ \|A\|_\infty = \max_{i=1,\ldots,n} \sum_{j=1}^{m} |a_{ij}|, \]
\[ \|A\|_2 = \left[ \rho(A^H A) \right]^{1/2} = \left[ \rho(A^T A) \right]^{1/2}, \]
\[ \|A\|_F = \left[ Tr(A^H A) \right]^{1/2} = \left[ Tr(A^T A) \right]^{1/2}. \]

**Exercise**

- Compute the Frobenius norms of the matrices
  \[
  \begin{pmatrix}
  1 & 1 \\
  1 & 0 \\
  3 & 2
  \end{pmatrix}
  \quad \begin{pmatrix}
  1 & 2 & -1 \\
  -1 & \sqrt{5} & 0 \\
  -1 & 1 & \sqrt{2}
  \end{pmatrix}
  \]

- Prove that the Frobenius norm is consistent [Hint: Use Cauchy-Schwartz]
- Define the 'vector 1-norm' of a matrix \( A \) as the 1-norm of the vector of stacked columns of \( A \). Is this norm a consistent matrix norm?
  [Hint: Result is true – Use Cauchy-Schwarz to prove it.]

- Compute the \( p \)-norm for \( p = 1, 2, \infty, F \) for the matrix
  \[
  A = \begin{pmatrix}
  0 & 2 \\
  0 & 1
  \end{pmatrix}
  \]

- Show that \( \rho(A) \leq \|A\| \) for any matrix norm.
Is $\rho(A)$ a norm?
1. $\rho(A) = \|A\|_2$ when $A$ is Hermitian ($A^H = A$). ➤ True for this particular case...
2. ... However, not true in general. For

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

we have $\rho(A) = 0$ while $A \neq 0$. Also, triangle inequality not satisfied for the pair $A$, and $B = A^T$. Indeed, $\rho(A + B) = 1$ while $\rho(A) + \rho(B) = 0.$

**A few properties of the 2-norm and the F-norm**

➤ Let $A = uv^T$. Then $\|A\|_2 = \|u\|_2\|v\|_2$

➤ Prove this result

➤ In this case $\|A\|_F = ??$

For any $A \in \mathbb{C}^{m \times n}$ and unitary matrix $Q \in \mathbb{C}^{m \times m}$ we have

$$\|QA\|_2 = \|A\|_2; \quad \|QA\|_F = \|A\|_F.$$

➤ Show that the result is true for any orthogonal matrix $Q$ ($Q$ has orthonomal columns), i.e., when $Q \in \mathbb{C}^{p \times m}$ with $p > m$

➤ Let $Q \in \mathbb{C}^{n \times n}$. Do we have $\|AQ\|_2 = \|A\|_2$? $\|AQ\|_F = \|A\|_F$? What if $Q \in \mathbb{C}^{n \times p}$, with $p < n$?