SPECIAL LINEAR SYSTEMS OF EQUATIONS

- **Symmetric positive definite matrices.**
- **The LDL^T decomposition; The Cholesky factorization**
- **Banded systems**

Positive-Definite Matrices

- A real matrix is said to be positive definite if

$$(Au, u) > 0 \text{ for all } u \neq 0 \; u \in \mathbb{R}^n$$

- Let A be a real positive definite matrix. Then there is a scalar $\alpha > 0$ such that

$$(Au, u) \geq \alpha \|u\|_2^2.$$

- Consider now the case of Symmetric Positive Definite (SPD) matrices.

- Consequence 1: A is nonsingular

- Consequence 2: the eigenvalues of A are (real) positive

A few properties of SPD matrices

➤ Diagonal entries of A are positive

➤ Recall: the k-th principal submatrix A_k is the $k \times k$ submatrix of A with entries a_{ij}, $1 \le i, j \le k$ (Matlab: $A(1:k, 1:k)$).

✍ Each A_k is SPD

✍ Consequence: $Det(A_k) > 0$ for $k = 1, \cdots, n$.

✍ For any $n \times k$ matrix X of rank k, the matrix $X^T A X$ is SPD.

➤ The mapping : $x, y \quad \rightarrow \quad (x, y)_A \equiv (Ax, y)$

defines a proper inner product on $\mathbb{R}^n$. The associated norm, denoted by $||.||_A$, is called the energy norm, or simply the A-norm:

$$||x||_A = (Ax, x)^{1/2} = \sqrt{x^T A x}$$

➤ Related measure in Machine Learning, Vision, Statistics: the Mahalanobis distance between two vectors:

$$d_A(x, y) = \|x - y\|_A = \sqrt{(x - y)^T A(x - y)}$$

Appropriate distance (measured in # standard deviations) if x is a sample generated by a Gaussian distribution with covariance matrix A and center y.

More terminology

- A matrix is Positive Semi-Definite if: $(Au, u) \geq 0$ *for all* $u \in \mathbb{R}^n$

- Eigenvalues of symmetric positive semi-definite matrices are real nonnegative, i.e., ...

- ... A can be singular [If not, A is SPD]

- A matrix is said to be Negative Definite if $-A$ is positive definite. Similar definition for Negative Semi-Definite

- A matrix that is neither positive semi-definite nor negative semi-definite is indefinite

✍ Show that if $A^T = A$ and $(Ax, x) = 0\ \forall x$ then $A = 0$

✍ Show: A is indefinite iff $\exists\ x, y : (Ax, x)(Ay, y) < 0$

The LDL^T and Cholesky factorizations

✍ The LU factorization of an SPD matrix A exists

➤ Let $A = LU$ and $D = diag(U)$ and set $M \equiv (D^{-1}U)^T$.

Then $$A = LU = LD(D^{-1}U) = LDM^T$$

➤ Both L and M are unit lower triangular

➤ Consider $L^{-1}AL^{-T} = DM^TL^{-T}$

➤ Matrix on the right is upper triangular. But it is also symmetric. Therefore $M^TL^{-T} = I$ and so $M = L$

➤ The diagonal entries of D are positive [Proof: consider $L^{-1}AL^{-T} = D$]. In the end:

$$A = LDL^T = GG^T \text{ where } G = LD^{1/2}$$

➤ Cholesky factorization is a specialization of the LU factorization for the SPD case. Several variants exist.

First algorithm: row-oriented LDLT

Adapted from Gaussian Elimination [Work only on upper triang. part]

```
1. For k = 1 : n − 1 Do:
2.     For i = k + 1 : n Do:
3.         piv := a(k, i)/a(k, k)
4.         a(i, i : n) := a(i, i : n) − piv ∗ a(k, i : n)
5.     End
6. End
```

➤ This will give the U matrix of the LU factorization. Therefore $D = diag(U)$, $L^T = D^{-1}U$.

Row-Cholesky (outer product form)

Scale the rows as the algorithm proceeds. Line 4 becomes

$$a(i,:) := a(i,:) - [a(k,i)/\sqrt{a(k,k)}] * \left[a(k,:)/\sqrt{a(k,k)}\right]$$

ALGORITHM : 1. *Outer product Cholesky*

1. For $k = 1 : n$ Do:
2. $\quad A(k, k:n) = A(k, k:n)/\sqrt{A(k,k)}$;
3. $\quad$ For $i := k+1 : n$ Do :
4. $\quad\quad A(i, i:n) = A(i, i:n) - A(k,i) * A(k, i:n);$
5. $\quad$ End
6. End

- Result: Upper triangular matrix U such $A = U^T U$.

Example:

$$A = \begin{pmatrix} 1 & -1 & 2 \\ -1 & 5 & 0 \\ 2 & 0 & 9 \end{pmatrix}$$

- ✍ Is A symmetric positive definite?
- ✍ What is the LDL^T factorization of A ?
- ✍ What is the Cholesky factorization of A ?

Column Cholesky. Let $A = GG^T$ with G = lower triangular. Then equate j-th columns:

$$a(i,j) = \sum_{k=1}^{j} g(j,k) g^T(k,i) \rightarrow$$

$$\begin{aligned} A(:,j) &= \sum_{k=1}^{j} G(j,k) G(:,k) \\ &= G(j,j) G(:,j) + \sum_{k=1}^{j-1} G(j,k) G(:,k) \rightarrow \end{aligned}$$

$$G(j,j) G(:,j) = A(:,j) - \sum_{k=1}^{j-1} G(j,k) G(:,k)$$

- Assume that first $j-1$ columns of G already known.
- Compute unscaled column-vector:

$$v = A(:,j) - \sum_{k=1}^{j-1} G(j,k) G(:,k)$$

- Notice that $v(j) \equiv G(j,j)^2$.
- Compute $\sqrt{v(j)}$ and scale v to get j-th column of G.

ALGORITHM : 2. *Column Cholesky*

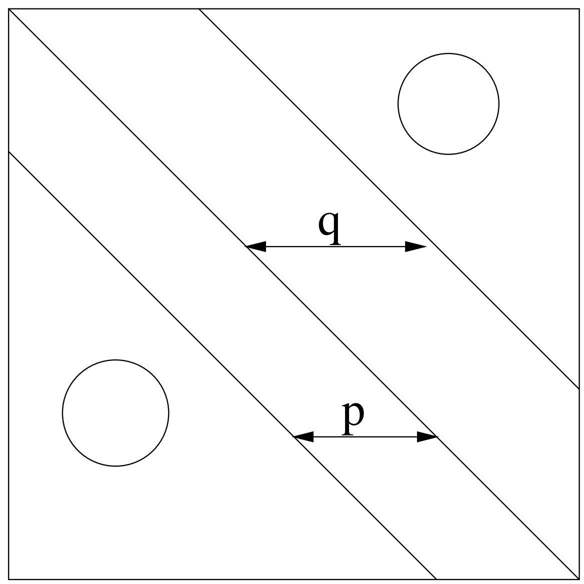
For $j = 1 : n$ *do*
    For $k = 1 : j - 1$ *do*
        $A(j:n, j) = A(j:n, j) - A(j, k) * A(j:n, k)$
    EndDo
    If $A(j, j) \leq 0$ *ExitError("Matrix not SPD")*
    $A(j, j) = \sqrt{A(j, j)}$
    $A(j+1:n, j) = A(j+1:n, j)/A(j, j)$
EndDo

Try algorithm on:

$$A = \begin{pmatrix} 1 & -1 & 2 \\ -1 & 5 & 0 \\ 2 & 0 & 9 \end{pmatrix}$$

Banded matrices

- Banded matrices arise in many applications
- A has upper bandwidth q if $a_{ij} = 0$ *for* $j - i > q$
- A has lower bandwidth p if $a_{ij} = 0$ *for* $i - j > p$



- Simplest case: tridiagonal ➤ $p = q = 1$.

➤ First observation: Gaussian elimination (no pivoting) preserves the initial banded form. Consider first step of Gaussian elimination:

```
For i = 2 : n Do:
    a_i1 := a_i1 / a_11 (pivots)
    For j := 2 : n Do :
        a_ij := a_ij − a_i1 * a_1j
    End
End
```

➤ If A has upper bandwidth q and lower bandwidth p then so is the resulting $[L/U]$ matrix. ➤ Band form is preserved (induction)

Operation count?

What happens when partial pivoting is used?

If A has lower bandwidth p, upper bandwidth q, and if Gaussian elimination with partial pivoting is used, then the resulting U has upper bandwidth $p+q$. L has at most $p+1$ nonzero elements per column (bandedness is lost).

➤ Simplest case: tridiagonal ➤ $p = q = 1$.

Example:

$$A = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 2 & 1 & 1 & 0 & 0 \\ 0 & 2 & 1 & 1 & 0 \\ 0 & 0 & 2 & 1 & 1 \\ 0 & 0 & 0 & 2 & 1 \end{pmatrix}$$