SPECIAL LINEAR SYSTEMS OF EQUATIONS

- **Symmetric positive definite matrices.**
- **The LDL^T decomposition; The Cholesky factorization**
- **Banded systems**

Positive-Definite Matrices

➤ A real matrix is said to be positive definite if

$$(Au, u) > 0 \text{ for all } u \neq 0 \; u \in \mathbb{R}^n$$

➤ Let A be a real positive definite matrix. Then there is a scalar $\alpha > 0$ such that

$$(Au, u) \geq \alpha \|u\|_2^2.$$

➤ Consider now the case of Symmetric Positive Definite (SPD) matrices.

➤ Consequence 1: A is nonsingular

➤ Consequence 2: the eigenvalues of A are (real) positive

A few properties of SPD matrices

➤ Diagonal entries of A are positive

➤ Recall: the k-th principal submatrix A_k is the $k \times k$ submatrix of A with entries a_{ij}, $1 \leq i, j \leq k$ (Matlab: $A(1:k, 1:k)$).

✍ Each A_k is SPD

✍ Consequence: $Det(A_k) > 0$ for $k = 1, \cdots, n$.

✍ For any $n \times k$ matrix X of rank k, the matrix $X^T A X$ is SPD.

➤ The mapping : $x, y \quad \rightarrow \quad (x, y)_A \equiv (Ax, y)$

defines a proper inner product on $\mathbb{R}^n$. The associated norm, denoted by $\|.\|_A$, is called the energy norm, or simply the A-norm:

$$\|x\|_A = (Ax, x)^{1/2} = \sqrt{x^T A x}$$

➤ Related measure in Machine Learning, Vision, Statistics: the Mahalanobis distance between two vectors:

$$d_A(x, y) = \|x - y\|_A = \sqrt{(x - y)^T A (x - y)}$$

Appropriate distance (measured in # standard deviations) if x is a sample generated by a Gaussian distribution with covariance matrix A and center y.

More terminology

- A matrix is Positive Semi-Definite if: $(Au, u) \geq 0$ *for all* $u \in \mathbb{R}^n$
- Eigenvalues of symmetric positive semi-definite matrices are real nonnegative, i.e., ...
- ... A can be singular [If not, A is SPD]
- A matrix is said to be Negative Definite if $-A$ is positive definite. Similar definition for Negative Semi-Definite
- A matrix that is neither positive semi-definite nor negative semi-definite is indefinite
- ✍ Show that if $A^T = A$ and $(Ax, x) = 0\ \forall x$ then $A = 0$
- ✍ Show: A is indefinite iff $\exists\ x, y : (Ax, x)(Ay, y) < 0$

The LDL^T and Cholesky factorizations

- ✍ The LU factorization of an SPD matrix A exists
- Let $A = LU$ and $D = diag(U)$ and set $M \equiv (D^{-1}U)^T$.

Then $A = LU = LD(D^{-1}U) = LDM^T$

- Both L and M are unit lower triangular
- Consider $L^{-1}AL^{-T} = DM^TL^{-T}$
- Matrix on the right is upper triangular. But it is also symmetric. Therefore $M^TL^{-T} = I$ and so $M = L$
- The diagonal entries of D are positive [Proof: consider $L^{-1}AL^{-T} = D$]. In the end:

$$A = LDL^T = GG^T \text{ where } G = LD^{1/2}$$

- Cholesky factorization is a specialization of the LU factorization for the SPD case. Several variants exist.

First algorithm: row-oriented LDLT

Adapted from Gaussian Elimination [Work only on upper triang. part]

1. For $k = 1 : n - 1$ Do:
2. For $i = k + 1 : n$ Do:
3. $piv := a(k, i)/a(k, k)$
4. $a(i, i : n) := a(i, i : n) - piv * a(k, i : n)$
5. End
6. End

- This will give the U matrix of the LU factorization. Therefore $D = diag(U)$, $L^T = D^{-1}U$.

Row-Cholesky (outer product form)

Scale the rows as the algorithm proceeds. Line 4 becomes

$$a(i,:) := a(i,:) - [a(k,i)/\sqrt{a(k,k)}] * \left[a(k,:)/\sqrt{a(k,k)}\right]$$

ALGORITHM : 1. *Outer product Cholesky*

1. *For* $k = 1 : n$ *Do:*
2. $A(k, k : n) = A(k, k : n)/\sqrt{A(k, k)}$;
3. *For* $i := k + 1 : n$ *Do :*
4. $A(i, i : n) = A(i, i : n) - A(k, i) * A(k, i : n)$;
5. *End*
6. *End*

- Result: Upper triangular matrix U such $A = U^TU$.

Example:

$$A = \begin{pmatrix} 1 & -1 & 2 \\ -1 & 5 & 0 \\ 2 & 0 & 9 \end{pmatrix}$$

- Is A symmetric positive definite?
- What is the LDL^T factorization of A ?
- What is the Cholesky factorization of A ?

Column Cholesky. Let $A = GG^T$ with G = lower triangular. Then equate j-th columns:

$$a(i,j) = \sum_{k=1}^{j} g(j,k) g^T(k,i) \rightarrow$$

$$A(:,j) = \sum_{k=1}^{j} G(j,k)G(:,k)$$

$$= G(j,j)G(:,j) + \sum_{k=1}^{j-1} G(j,k)G(:,k) \rightarrow$$

$$G(j,j)G(:,j) = A(:,j) - \sum_{k=1}^{j-1} G(j,k)G(:,k)$$

- Assume that first $j-1$ columns of G already known.
- Compute unscaled column-vector:

$$v = A(:,j) - \sum_{k=1}^{j-1} G(j,k)G(:,k)$$

- Notice that $v(j) \equiv G(j,j)^2$.
- Compute $\sqrt{v(j)}$ and scale v to get j-th column of G.

ALGORITHM : 2. *Column Cholesky*

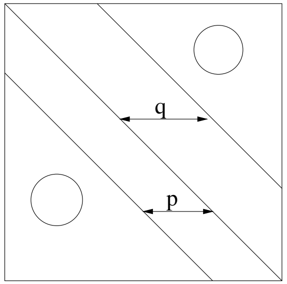
1. *For* $j = 1 : n$ *do*
2. *For* $k = 1 : j-1$ *do*
3. $A(j:n,j) = A(j:n,j) - A(j,k) * A(j:n,k)$
4. *EndDo*
5. *If* $A(j,j) \le 0$ *ExitError("Matrix not SPD")*
6. $A(j,j) = \sqrt{A(j,j)}$
7. $A(j+1:n,j) = A(j+1:n,j)/A(j,j)$
8. *EndDo*

- Try algorithm on:

$$A = \begin{pmatrix} 1 & -1 & 2 \\ -1 & 5 & 0 \\ 2 & 0 & 9 \end{pmatrix}$$

Banded matrices

- Banded matrices arise in many applications
- A has upper bandwidth q if $a_{ij} = 0$ *for* $j - i > q$
- A has lower bandwidth p if $a_{ij} = 0$ *for* $i - j > p$



- Simplest case: tridiagonal ➤ $p = q = 1$.

- First observation: Gaussian elimination (no pivoting) preserves the initial banded form. Consider first step of Gaussian elimination:

2. For $i = 2 : n$ Do:
3. $a_{i1} := a_{i1}/a_{11}$ (pivots)
4. For $j := 2 : n$ Do :
5. $a_{ij} := a_{ij} - a_{i1} * a_{1j}$
6. End
7. End

- If A has upper bandwidth q and lower bandwidth p then so is the resulting $[L/U]$ matrix. ➤ Band form is preserved (induction)

✍ Operation count?

What happens when partial pivoting is used?

If A has lower bandwidth p, upper bandwidth q, and if Gaussian elimination with partial pivoting is used, then the resulting U has upper bandwidth $p + q$. L has at most $p + 1$ nonzero elements per column (bandedness is lost).

- Simplest case: tridiagonal ➤ $p = q = 1$.

Example:

$$A = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 2 & 1 & 1 & 0 & 0 \\ 0 & 2 & 1 & 1 & 0 \\ 0 & 0 & 2 & 1 & 1 \\ 0 & 0 & 0 & 2 & 1 \end{pmatrix}$$