Orthogonal projectors and the URV decomposition

- Orthogonal subspaces;
- Orthogonal projectors; Orthogonal decomposition;
- The URV decomposition
- Introduction to the Singular Value Decomposition
Orthogonal projectors and subspaces

Notation: Given a subspace $\mathcal{X}$ of $\mathbb{R}^m$ define

$$\mathcal{X}^\perp = \{ y \mid y \perp x, \ \forall \ x \in \mathcal{X} \}$$

Let $Q = [q_1, \ldots, q_r]$ an orthonormal basis of $\mathcal{X}$

How would you obtain such a basis?

Then define orthogonal projector $P = QQ^T$

Properties

(a) $P^2 = P$ \quad (b) $(I - P)^2 = I - P$

(c) $\text{Ran}(P) = \mathcal{X}$ \quad (d) $\text{Null}(P) = \mathcal{X}^\perp$

(e) $\text{Ran}(I - P) = \text{Null}(P) = \mathcal{X}^\perp$

Note that (b) means that $I - P$ is also a projector
Proof. (a), (b) are trivial

(c): Clearly \( \text{Ran}(P) = \{x | x = QQ^Ty, y \in \mathbb{R}^m\} \subseteq \mathcal{X} \). Any \( x \in \mathcal{X} \) is of the form \( x = Qy, y \in \mathbb{R}^m \). Take \( Px = QQ^T(Qy) = Qy = x \). Since \( x = Px, x \in \text{Ran}(P) \). So \( \mathcal{X} \subseteq \text{Ran}(P) \). In the end \( \mathcal{X} = \text{Ran}(P) \).

(d): \( x \in \mathcal{X}^\perp \iff (x, y) = 0, \forall y \in \mathcal{X} \iff (x, Qz) = 0, \forall z \in \mathbb{R}^r \iff (Q^Tx, z) = 0, \forall z \in \mathbb{R}^r \iff Q^Tx = 0 \iff QQ^Tx = 0 \iff Px = 0 \).

(e): Need to show inclusion both ways.

- \( x \in \text{Null}(P) \iff Px = 0 \iff (I - P)x = x \rightarrow x \in \text{Ran}(I - P) \)
- \( x \in \text{Ran}(I - P) \iff \exists y \in \mathbb{R}^m | x = (I - P)y \rightarrow Px = P(I - P)y = 0 \rightarrow x \in \text{Null}(P) \).
Result: Any \( x \in \mathbb{R}^m \) can be written in a unique way as

\[ x = x_1 + x_2, \quad x_1 \in X, \quad x_2 \in X^\perp \]

Proof: Just set \( x_1 = Px, \quad x_2 = (I - P)x \)

Note: \( X \cap X^\perp = \{0\} \)

Therefore: \( \mathbb{R}^m = X \oplus X^\perp \)

Called the **Orthogonal Decomposition**
In other words $\mathbb{R}^m = PR^m \oplus (I - P)\mathbb{R}^m$ or:

$\mathbb{R}^m = \text{Ran}(P) \oplus \text{Ran}(I - P)$ or:

$\mathbb{R}^m = \text{Ran}(P) \oplus \text{Null}(P)$ or:

$\mathbb{R}^m = \text{Ran}(P) \oplus \text{Ran}(P)^\perp$

Can complete basis $\{q_1, \cdots, q_r\}$ into orthonormal basis of $\mathbb{R}^m$,

$q_{r+1}, \cdots, q_m$

$\{q_{r+1}, \cdots, q_m\} =$ basis of $\mathcal{X}^\perp$. $\rightarrow \text{dim}(\mathcal{X}^\perp) = m - r.$
Let \( A \in \mathbb{R}^{m \times n} \) and consider \( \text{Ran}(A)^\perp \)

**Property 1:** \( \text{Ran}(A)^\perp = \text{Null}(A^T) \)

*Proof:* \( x \in \text{Ran}(A)^\perp \) iff \( (Ay, x) = 0 \) for all \( y \) iff \( (y, A^T x) = 0 \) for all \( y \) ...

**Property 2:** \( \text{Ran}(A^T) = \text{Null}(A)^\perp \)

 glide > Take \( X = \text{Ran}(A) \) in orthogonal decomposition. ➤ Result:

\[
\mathbb{R}^m = \text{Ran}(A) \oplus \text{Null}(A^T) \quad \text{4 fundamental subspaces} \\
\mathbb{R}^n = \text{Ran}(A^T) \oplus \text{Null}(A) \\
\text{Ran}(A) \quad \text{Null}(A^T), \quad \text{Ran}(A^T) \quad \text{Null}(A) 
\]
Express the above with bases for $\mathbb{R}^m$:

$$\begin{bmatrix} u_1, u_2, \cdots, u_r, u_{r+1}, u_{r+2}, \cdots, u_m \end{bmatrix}_{\text{Ran}(A)} \quad \begin{bmatrix} u_{r+1}, u_{r+2}, \cdots, u_m \end{bmatrix}_{\text{Null}(A^T)}$$

and for $\mathbb{R}^n$:

$$\begin{bmatrix} v_1, v_2, \cdots, v_r, v_{r+1}, v_{r+2}, \cdots, v_n \end{bmatrix}_{\text{Ran}(A^T)} \quad \begin{bmatrix} v_{r+1}, v_{r+2}, \cdots, v_n \end{bmatrix}_{\text{Null}(A)}$$

Observe $u_i^T A v_j = 0$ for $i > r$ or $j > r$. Therefore

$$U^T A V = R = \begin{pmatrix} C & 0 \\ 0 & 0 \end{pmatrix}_{m \times n} \quad C \in \mathbb{R}^{r \times r} \quad \rightarrow$$

$$A = U R V^T$$

General class of URV decompositions
Far from unique.

Show how you can get a decomposition in which $C$ is lower (or upper) triangular, from the above factorization.

Can select decomposition so that $R$ is upper triangular $\rightarrow$ URV decomposition.

Can select decomposition so that $R$ is lower triangular $\rightarrow$ ULV decomposition.

SVD = special case of URV where $R = \text{diagonal}$

How can you get the ULV decomposition by using only the Householder QR factorization (possibly with pivoting)? [Hint: you must use Householder twice]