Please do all problems; we will grade a subset of the assigned problems (same subset for everyone).

Please follow all of the instructions given in the handout for Homework 1.

Note: The emphasis in this assignment is on proofs, and these carry most of the points. Therefore, you should ensure that your proofs are written carefully and clearly.

1. (14 points) Let \( P = \{p_1 < p_2 < \cdots < p_n\} \) be a set of points on the real line; the distance between consecutive points can be arbitrary. We are given an unlimited supply of closed intervals of length 1 and would like to determine the smallest number of such intervals to place on the real line so that each point of \( P \) is contained in some interval and the intervals are non-overlapping. (By a “closed interval” we mean that the endpoints are included in the interval and by “intervals are non-overlapping” we mean that the intervals have no point in common.)

Describe briefly, in words, a greedy algorithm for this problem (pseudocode is not required). Prove it correct using the 2-step method, i.e., state and establish carefully the greedy choice and the optimal substructure properties.

2. (14 points) Let \( A \) and \( B \) be sequences of \( n \) positive integers each. You are allowed to re-order \( A \) and \( B \) as you wish. Let \( A = a_1, a_2, \ldots, a_n \) and \( B = b_1, b_2, \ldots, b_n \) after the re-ordering. The goal is to come up with an ordering which maximizes \( \prod_{i=1}^{n} a_i b_i \).

Describe briefly, in words, a greedy algorithm for this problem (pseudocode is not required). Prove it correct using the 2-step method, i.e., state and establish carefully the greedy choice and the optimal substructure properties.

3. (12 points) Ex. 23.1-5, p. 629.

4. (12 points) Ex. 23.1-6, p. 630.

5. (12 points) Prove that the set system \( M' = (S', \mathcal{I}') \) in Lemma 16.10, p. 442 is indeed a matroid, by showing that the defining properties of a matroid hold. (The book assumes implicitly that \( M' \) is a matroid.) Take \( S' = \{y \in S - \{x\} \mid \{x, y\} \in \mathcal{I}\} \), rather than as defined in the lemma.

6. (12 points) Let \( M = (S, \mathcal{I}) \) be a matroid. Let \( H \) be any given subset of \( S \) and define \( \mathcal{I}' = \{C \mid C \in \mathcal{I} \text{ and } C \cap H = \emptyset\} \). (Here \( \emptyset \) denotes the empty set.) Prove that \( M' = (S, \mathcal{I}') \) is a matroid by showing that the defining properties of a matroid hold.

7. (12 points) Let \( T \) be an \( n \)-node binary search tree of height \( h \). Let \( k \) be some integer, where \( 0 \leq k \leq n \). Suppose that we start at some node of \( T \) and make \( k \) successive calls to the routine TREE-SUCCESSOR. Prove that the total time taken for all calls is \( O(k + h) \) no matter which node we start at. (The calls are successive in the sense that each call after the first is made from the node where the previous call terminates.)
A careful answer is expected. Note that an $O(kh)$ time bound is easy to establish but is not of interest in this problem (and will earn no credit).

*Hint:* Consider separately the upward and downward traversals of left branches (and, similarly, of right branches) and obtain upper bounds on the number of traversals of each type.