2. If $A \in \mathbb{R}^{m \times n}$ what are the dimensions of $A^\dagger$?, $A^\dagger A$?, $AA^\dagger$?

**Solution:** The dimension of $A^\dagger A$ is $n \times m$ and so $A^\dagger A$? is of size $n \times n$. Similarly, $AA^\dagger$ is of size $m \times m$.

3. Show that $A^\dagger A$ is an orthogonal projector. What are its range and null-space?

**Solution:** One way to do this is to use the rank-one expansion: $A = \sum \sigma_i u_i v_i^T$. Then $A^\dagger = \sum \frac{1}{\sigma_i} v_i u_i^T$ and therefore,

$$A^\dagger A = \left[ \sum_{i=1}^r \frac{1}{\sigma_i} v_i u_i^T \right] \times \left[ \sum_{j=1}^r \sigma_j u_j v_j^T \right] = \sum_{j=1}^r v_j v_j^T$$

which is a projector.
4. Consider the matrix:

\[ A = \begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & 0 & -2 & 1 \end{pmatrix} \]

- Compute the singular value decomposition of \( A \)

Find the SVD of \( A \) ...

**Solution:** The nonzero singular values of \( A \) are the square roots of the eigenvalues of

\[ A A^T = \begin{pmatrix} 5 & -4 \\ -4 & 5 \end{pmatrix} \]

These eigenvalues are \( 5 \pm 4 \) and so \( \sigma_1 = 3, \sigma_2 = 1 \).

The matrix \( U \) of the left singular vectors is the matrix

\[ U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \]

If \( A = U \Sigma V^T \), then \( U' \ast A = \Sigma V^T \). Therefore to get \( V \) we use the relation: \( V^T = \)
\( \Sigma^{-1} \ast U' \ast A \). We have

\[
U' \ast A = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 4 & -1 \\ 1 & 0 & 0 & 1 \end{pmatrix} \rightarrow V^T = \frac{1}{\sqrt{2}} \begin{pmatrix} 1/3 & 0 & 4/3 & -1/3 \\ 1 & 0 & 0 & 1 \end{pmatrix} \rightarrow
\]

- Find the matrix \( B \) of rank 1 which is the closest to \( A \) in 2-norm sense.

**Solution:** This is obtained by setting \( \sigma_2 \) to zero in the SVD - or - equivalently as \( B = \sigma_1 u_1 v_1^T \).

You will find

\[
B = \begin{pmatrix} 1/2 & 0 & 2 & -1/2 \\ -1/2 & 0 & -2 & 1/2 \end{pmatrix}
\]