Use the min-max theorem to show that $\|A\|_2 = \sigma_1(A)$ - the largest singular value of $A$.

**Solution:** This comes from the fact that:

$$
\|A\|_2^2 = \max_{x \neq 0} \frac{\|Ax\|_2^2}{\|x\|_2^2} = \max_{x \neq 0} \frac{(Ax, Ax)}{(x, x)} = \max_{x \neq 0} \frac{(A^T Ax, A)}{(x, x)} = \lambda_{max}(A^T A) = \sigma_1^2
$$

Suppose that $A = LDL^T$ where $L$ is unit lower triangular, and $D$ diagonal. How many negative eigenvalues does $A$ have?
Solution: It has as many negative eigenvalues as there are negative entries in $D$.

Assume that $A$ is tridiagonal. How many operations are required to determine the number of negative eigenvalues of $A$?

Solution: The rough answer is $O(n)$ – because an LU (and therefore LDLT) factorization costs $O(n)$. Based on doing the LU factorization of a triagonal matrix, a more accurate answer is $3n$ operations.

Devise an algorithm based on the inertia theorem to compute the $i$-th eigenvalue of a tridiagonal matrix.

Solution: Here is a matlab script:

```matlab
function [sigma] = bisect(d, b, i, tol)
    %% function [sigma] = bisect(d, b, i, tol)
    %% d = diagonal of T
    %% b = co-diagonal
    %% i = compute i-th eigenvalue
    %% tol = tolerance used for stopping
    b(1) = 0;
    n = length(d);
    %%------------------------- guershgorin
```
What is the inertia of the matrix

\[
\begin{pmatrix}
I & F \\
F^T & 0
\end{pmatrix}
\]

where \( F \) is \( m \times n \), with \( n < m \), and of full rank?
Solution: We start with
\[
\begin{pmatrix}
I & F \\
F^T & 0
\end{pmatrix}
= 
\begin{pmatrix}
I & 0 \\
F^T & I
\end{pmatrix}
\begin{pmatrix}
I & F \\
0 & -F^TF
\end{pmatrix}
= 
\begin{pmatrix}
I & 0 \\
F^T & I
\end{pmatrix}
\begin{pmatrix}
I & 0 \\
0 & -F^TF
\end{pmatrix}
\begin{pmatrix}
I & F \\
0 & I
\end{pmatrix}
= 
\begin{pmatrix}
I & 0 \\
F^T & I
\end{pmatrix}
\begin{pmatrix}
I & 0 \\
0 & -F^TF
\end{pmatrix}
\begin{pmatrix}
I & 0 \\
F^T & I
\end{pmatrix}
^T
\]

This is of the form \( XDX^T \) where \( X \) is invertible. Therefore the inertia is the same as that of the block diagonal matrix which is: \( m \) positive eigenvalues (block \( I \)) and \( n \) negative eigenvalues since \(-F^TF\) is \( n \times n \) and negative definite.

\[ \square \]

Let \( \| A_O \|_I = \max_{i \neq j} |a_{ij}| \). Show that
\[
\| A_O \|_F \leq \sqrt{n(n-1)} \| A_O \|_I
\]
Solution: This is straightforward:

\[ \| A_O \|_F^2 = \sum_{i \neq j} |a_{ij}|^2 \leq n(n-1) \max_{i \neq j} |a_{ij}|^2 = n(n-1) \| A_O \|_I^2. \]

Use this to show convergence in the case when largest entry is zeroed at each step.

Solution: If we call \( B_k \) the matrix \( A_O \) after each rotation then we have according to result in the previous page and using the previous exercise:

\[
\| B_{k+1} \|_F^2 = \| B_k \|_F^2 - 2a_{pq}^2 \\
= \| B_k \|_F^2 - 2\| B_k \|_I^2 \\
\leq \| B_k \|_F^2 - \frac{2}{n(n+1)} \| B_k \|_F^2 \\
= \left[ 1 - \frac{2}{n(n+1)} \right] \| B_k \|_F^2
\]

which shows that the norm will be decreasing by factor less than a constant that is less than one - therefore it converges to zero. \( \square \)