Prove that Gram-Schmidt can be completed iff the $x_i$’s are linearly independent.

Solution:

We will show that Gram-Schmidt breaks down iff the $x_i$’s are linearly dependent.

The only way in which GS can break down is if $r_{jj} = \|\hat{q}\|_2$ in line 7 is zero.

The main observation is that the vector $\hat{q}$ at the end of the loop starting in line 4 (ending in line 6) is a linear combination of the $x_i$’s for $i = 1, \ldots, j - 1$. [simple proof by induction – omitted.] □

Cost of Gram-Schmidt?

Solution: Step $j$ of the algorithm costs: $(j - 1) \times 2m$ operations for line 3, + $(j - 1) \times 2m$ operations for loop in line 4 + $3m$ operations in Lines 7 and 8 together. Total for step $j =$
\( c_j = (4j - 1)m \). Total over the \( n \) columns = \( T(n) = (2n^2 + n)m \approx 2n^2m \).

Note: this is linear in \( m \) (number of rows) and quadratic in \( n \) (number of columns).

What is the cost of solving a linear system with the QR factorization?

**Solution:** According to the previous question we have a cost of \( 2n^3 \) for the factorization (since \( m = n \)), to which we need to add the cost of solving a triangular solve \( O(n^2) \) and the cost for computing \( Q^Tb \) which is again \( O(n^2) \). In the end the cost is dominated by the QR factorization which is \( 2n^3 \). This is 3 times more expensive than GE.