How would you get an orthonormal basis of $X$?

**Solution:** Just take any basis and orthonormalize it with Gram-Schmidt or Householder QR.

Show how you can get a decomposition in which $C$ is lower (or upper) triangular, from the above factorization.

**Solution:** You first get any factorization in the form shown in Page 9-7 – Then to get an upper triangular $C$ you use the QR factorization $C = QR$. Then $U$ is replaced by

$$U_{new} = U \times \begin{pmatrix} Q & 0 \\ 0 & I \end{pmatrix}$$

and $C$ is replaced by $R$. To get a lower triangular $C$ you can use the same trick applied to $A^T$ and transpose the final result.
How can you get the ULV decomposition by using only the Householder QR factorization (possibly with pivoting)?

**Solution:** You first get the Householder QR factorization $A = Q_1 R_1$ of the matrix $A$. The second step is to perform a Householder QR factorization of the matrix $R_1^T$, so you will get: $R_1^T = Q_2 R_2$. The final step is to write:

$$A = Q_1 \ast R_2^T \ast Q_2^T \equiv U R V^T$$

where $U = Q_1 \in \mathbb{R}^{m \times m}; V = Q_2 \in \mathbb{R}^{n \times n}; R = R_2 \in \mathbb{R}^{m \times n}$.

In the proof of the SVD decomposition, define $U, V$ as single Householder reflectors.

**Solution:** We deal with $U$ only [proceed similarly with $V$]. We need a matrix $P = I - 2ww^T$ such that the first column of $A$ is $u_1$ and all columns are orthonormal. The second requirement is satisfied by default since $P$ is unitary. Note that what if $P$ is available we will have $Pu_1 = e_1$ because $P^2 = I$. Therefore, the wanted $w$ is simply the vector that transforms the vector $u_1$ into $\alpha e_1$. ...
How can you obtain the thin SVD from the QR factorization of $A$ and the SVD of an $n \times n$ matrix?

**Solution:** We first get the thin QR factorization of $A$, namely $A = QR$ where $Q \in \mathbb{R}^{m \times n}$ and $R \in \mathbb{R}^{n \times n}$; Then we can get the SVD $R = U_R \Sigma_R V_R^T$ of $R$ and this yields:

$$A = Q \times U_R \Sigma_R V_R^T \rightarrow A = U \Sigma V^T, \quad \text{with} \quad U = Q \times U_R; \quad \Sigma = \Sigma_R; \quad V = V_R.$$