1. Let $X$ be an $m \times n$ matrix, with $m \geq n$, that is of full rank. Show that $X^TX$ is nonsingular. [Hint: By making judicious use of inner products, show that $X^TY = 0$ implies that $Xy = 0$ which in turn implies that $y = 0$.]

2. Show that $A \in \mathbb{R}^{m \times n}$ is of rank 1 if and only if there are two vectors $u \in \mathbb{R}^m$ and $v \in \mathbb{R}^n$ such that $A = uv^T$. Generalize this to the case where the rank is $p > 1$ (with $p \leq \min\{m, n\}$) i.e., show that a matrix $A \in \mathbb{R}^{m \times n}$ is of rank $p$ if and only if there exists a full-rank matrix $X \in \mathbb{R}^{m \times p}$ and a full-rank matrix $Y \in \mathbb{R}^{n \times p}$ such that $A = XY^T$.

3. Let $A$ be an $n \times n$ matrix whose only nonzero entries are in the first column and first row (i.e., $a_{i,j} = 0$ when $i > 1$ and $j > 1$). (a) Show that $A$ is of rank $\leq 2$. When is the rank less than 2? (b) Assume that in addition $A$ is symmetric and that $a_{11} = 1$. Show that there exist two vectors $u$ and $v$ such that $A = uu^T - vv^T$.

4. Let $T$ be a symmetric Toeplitz matrix $T = [t_{i-j}]_{i,j=1}^n$ with $t_0 = 1$ and define the $n \times n$ lower triangular shift matrix $Z$:

$$Z = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 1 & 0 \end{pmatrix}$$

(a) Show that the matrix $D = T - ZTZ^T$ has rank 2. [Hints: For a matrix $X$ what are the columns of $XZ^T$? What are the rows of $ZX$? Then use the result of the previous exercise.] (b) Show that $D$ can be written as $D = uu^T - vv^T$ for certain vectors $u, v$ to be specified. (Note: $D$ is called the displacement of $T$ with respect to $Z$ and the rank of $D$ is the displacement rank of $T$. Matrices with low displacement ranks have been extensively studied.)

5. Are the following functions from $\mathbb{R}^n$ to $\mathbb{R}$ vector norms? (prove or disprove).

$$(a) : N(x) = \sum_{i=1}^{n} \frac{x^2_i}{2}; \quad (b) : N(x) = \left[ \sum_{i=1}^{n} |x_i|^{1/3} \right]^3; \quad (c) : N(x) = \left[ \sum_{i=1}^{n} |x_i| \right]^3;$$

6. Let $\| \cdot \|$ be a norm in $\mathbb{R}^n$. Define:

$$\|x\|' = \sup\{u^T x : u \in \mathbb{R}^n, \|u\| = 1\}$$

(a) Prove that this equation defines a norm (called the “dual norm” of $\| \cdot \|$).

(b) Show that for all $x, y \in \mathbb{R}^n$ we have

$$|x^Ty| \leq \|x\| \|y\|'$$

(c) What is the dual norm of $\| \cdot \|_1$?
7. (a) Calculate $\|A\|_1, \|A\|_\infty$ for the matrix:

$$A = \begin{pmatrix}
1 & 6 & 0 \\
6 & -1 & 3 \\
-2 & 3 & 5
\end{pmatrix}$$

(b) Among all vectors $x$ satisfying $\|x\|_\infty \leq 1$ find one for which $\|Ax\|_\infty$ is the largest possible.

(c) Among all vectors $x$ satisfying $\|x\|_1 \leq 1$ find one for which $\|Ax\|_1$ is the largest possible.

(d) (Use matlab) Calculate the 2-norm of $A$. Among all vectors $x$ satisfying $\|x\|_2 \leq 1$ find one for which $\|Ax\|_2$ is the largest possible.

8. Let $B$ be unitary of dimension $n \times n$ and $A$ be an arbitrary matrix of dimension $n \times n$. Show that $\|AB\|_F = \|A\|_F$. Assume now that $B$ is $m \times n$ and orthogonal ($m > n$). Show that $\|BA\|_F = \|A\|_F$.

9. [Matlab Exercise]
   a. Compute the 1-norm, the 2-norm, the infinity norm and the Frobenius norm of the matrix:

   $$\begin{pmatrix}
   -1 & 2 & 1 \\
   0 & 0 & 0 \\
   -2 & 3 & 0 \\
   1 & -1 & 1 \\
   2 & -1 & 4
   \end{pmatrix}$$

   b. Find the eigenvalues and the spectral radius of $A(2 : 4,:)$

   c. Find the singular values of $A$. What is the nuclear norm of $A$? What is its Schatten 3-norm? From the singular values what can you say about the determinant of $A^T A$?

   d. Using the result of Question 1, and the information on the determinant of question (c) above show that the matrix $A$ does not have full rank. Use matlab’s ‘rank’ function to determine the rank of $A$.

   e. Explore the “reduced row echelon form” function of matlab called rref. Once you understand what the rref function does, use it to find the RREF form of $A$. Can you explain in words what was done to obtain this form? [recall Gauss-Jordan elimination].