1. Consider the following two algorithms to compute the function \( f(x) = (e^x - 1)/x = \sum_{i=0}^{\infty} x^i/(i+1)! \) which arises in many applications:

**Algorithm 1**

```matlab
if x == 0
    f = 1
else
    f = (exp(x)-1)/x;
end
```

**Algorithm 2**

```matlab
y = exp(x)
if y == 1
    f = 1;
else
    f = (y-1)/log(y);
end
```

Using matlab, tabulate results of both algorithms for \( x = 10^{-5}, 10^{-6}, 10^{-7}, \ldots, 10^{-16} \). In a third column show also the values of \( 1 + x/2 + x^2/6 \) which are obtained from the Taylor series expansion (second order accurate). Use 'format long' to display your results. Discuss and explain what you observe [Hint: in first algorithm you need to show that a cancellation error is amplified. For second algorithm note that the first step computes \( f_l(y) = e^x(1 + \delta) \) which you can write as \( \hat{y} = e^x \). How accurate is the rest of the computation? What is \( x - \hat{x} \) approximately?]

2. Let \( A \) and \( B \) two nonsingular matrices of size \( n \times n \). Show that

\[
fl(AB) = (A + E_A)B \quad \text{where} \quad |E_A| \leq \gamma_n |A| |B| |B^{-1}|
\]

(\( \gamma_n \) defined in the notes) and derive a corresponding bound in which \( B \) is perturbed. [This result shows the limitation of backward error analysis. In this case it is clear that Forward error analysis yields a ‘cleaner’ result].

3. Assume the IEEE standard for floating point arithmetic (see relevant pages in the notes that supplement set number # 4). If you use the matlab command 'num2hex' you will find that the IEEE hex representation of the number '-6.0' (negative 6) is

'c018000000000000000000000000000000000000000000000000000000000000'

The 'exponent+sign' part is 'c01' and the mantissa is 8 followed by 12 zeros (in HEX). Explain why you obtain this representation. [Hint: In binary the very first bit is one which indicates a negative sign].

4. Using matlab, plot in a logarithmic scale the condition numbers \( \kappa_2(H_n) \) for \( n = 3 : 12 \), where \( H_n \) is the Hilbert matrix of dimension \( n \). [Note: the matlab command hilb(n) generates the \( n \)-th Hilbert matrix.] Based on the plot give an approximate expression for the condition number \( \kappa_2(H_n) \) as a function of \( n \).

5. Consider the matrix \( A \) shown on the right, where \( I \) represents the \( n \times n \) identity.

(a) What is the inverse of \( A \)?
(b) Show that \( \kappa_F(A) = 2n + \|Z\|_F^2 \).
(c) Express the 1-norm condition number of \( A \) in terms of the 1-norm of \( Z \).
(d) Express the \( \infty \)-norm condition number of \( A \) in terms of the \( \infty \)-norm of \( Z \).
6. Show that \( \kappa(AB) \leq \kappa(A)\kappa(B) \). Is it true in general that \( \kappa(A) = \kappa(A^T) \)? Show that \( \kappa_2(A) = \kappa_2(A^T) \) and \( \kappa_2(A^T A) = \kappa_2(A)^2 \).

7. Consider the following two systems.

\[
Ax \equiv \begin{pmatrix} 1 & 1 & -1 \\ 1 & 2 & 0.01 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -1 \\ 1.01 \\ 2.0 \end{pmatrix} \quad \tilde{Ay} \equiv \begin{pmatrix} 1 & 1 & -1 \\ 1.0001 & 2 & 0.01 \\ 0.0 & 1 & 1.0001 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} -1 \\ 1.01 \\ 2.0 \end{pmatrix}
\]

The solution of the first system is \( x_1 = -1; x_2 = x_3 = 1 \). (a) Using matlab compute the solution \( y \) to the second system and compute also \( \kappa_\infty(A) \). (b) Then compute \( \|x - y\|_\infty / \|x\|_\infty \) and an upper bound for it obtained using the condition number.

8. (a) Obtain a lower bound for \( \kappa_\infty(A) \) (without computing \( A^{-1} \)) for the matrix

\[
A = \begin{pmatrix} 1 & 2 & 3 \\ -1 & 1 & -2 \\ 0.001 & 3 & 1 \end{pmatrix}
\]

[Hint: \( A \) is close to a singular matrix]

(b) Using matlab get the (exact) condition number \( \kappa_\infty(A) \). Let \( B \) be the matrix obtained from \( A \) by multiplying its 3rd row by 1000. Get the condition number \( \kappa_\infty(B) \).

9. Show (without computing eigenvalues) that the following symmetric matrices are not positive definite:

\[
A = \begin{pmatrix} 0 & 3 & 1 \\ 3 & 2 & 0 \\ 1 & 0 & 2 \end{pmatrix} \quad B = \begin{pmatrix} 2 & 3 & 1 \\ 3 & 2 & 0 \\ 1 & 0 & 2 \end{pmatrix} \quad C = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 2 \end{pmatrix}
\]

10. Let \( A \in \mathbb{R}^{n \times n} \) be a symmetric positive definite matrix.

(a) Show that \( |a_{ij}| \leq \sqrt{a_{ii}a_{jj}} \)

(b) Show also that \( |a_{ij}| \leq (a_{ii} + a_{jj})/2 \)