1. Consider the matrix

\[
A = \begin{pmatrix}
4 & -2 & 4 \\
-2 & 2 & -2 \\
4 & -2 & 8
\end{pmatrix}
\]

(a) Find the LU factorization of \( A \) (no pivoting);
(b) Find its LDLT factorization from the previous question;
(c) Find its Cholesky factorization (again using (a)).
(d) Suppose you want to decrease \( a_{33} \) from its value of 8, so that the modified \( A \) remains positive definite. What is the lower limit for \( a_{33} \) (also say if the limit is itself acceptable).

2. The Hald cement data is used in several books and papers as an example of regression analysis.

The right-hand side is the heat evolved in cement during hardening, and the explanatory variables are four different ingredients \( x_1, \ldots, x_4 \) of the mix. The right-hand side \( b \) and the matrix \( A = [a_1, \ldots, a_4] \) corresponding to 13 measurements, are available from the script \texttt{Hald.m} which you will find in the matlab section of the class web-site. [Incidentally this is available if you have access to the Statistics and Machine Learning Toolbox – simply type \texttt{load hald}.] In what follows, \( e \) is the vector of all ones. We modify \( A \) by adding to it the column \( e \) as a first column (\( A = [\text{ones}(13, 1), A] \) in Matlab). This corresponds to adding a constant to the regression model: so the model is is of the form \( b = x_0 + a_1x_1 + \cdots + a_4x_4 \).

(a) Solve the least squares problem \( \min \| b - Ax \|_2 \) by the method of normal equations. Obtain \( \kappa_2(A) \).

(b) We now show how to get rid of the constant unknown from the system. Write \( x = \begin{pmatrix} \xi \\ y \end{pmatrix} \) where \( \xi \) is a scalar, and show how to eliminate \( \xi \) from the system [Hint: Start with the orthogonality conditions for optimality but restrict these to only one condition involving the vector \( e \)]. The resulting problem is now a least-squares problem of the form \( \min \| By - c \|_2 \) involving only the \( y \) vector. What are \( B \) and \( c \)? What is the condition number of this reduced problem?

(c) Continued from (b) How can you interpret \( c \) relative to \( b \) and \( B \) relative to \( X \)? [Hint: What is \( e^T B \) ? What is \( e^T c \) ?]

3. The purpose of this exercise is to test 3 different ways of computing the QR factorization of a matrix \( A \)

(a) The classical Gram-Schmidt algorithm
(b) The modified Gram-Schmidt algorithm
(c) The Cholesky factorization of \( A^T A \)

Explain how the Cholesky factorization of \( A^T A \) can be used. In the following you should use the script \texttt{cholR} that is posted (not the \texttt{chol} function from matlab). You can use \texttt{inv} to invert triangular matrices.

A data set is posted on the class web-site (see the matlab page). Write a script which loads the matrix and then for each of the three methods above compute the \( Q \) and \( R \) factors and the error measures

\[
\| A - Q \ast R \|_2, \quad \| I - Q^T \ast Q \|_2
\]
Present your result in the form of a table and comment on them.

4. Let $A$ be an $m \times n$ real matrix of full rank. Prove that if $\|Au\|_2 = \|u\|_2$ for all vectors $u \in \mathbb{R}^n$ then $A$ is an orthogonal matrix (i.e., $A^TA = I$). [Hint: There are several ways of doing this. One of them is to exploit the exercise 4 of Lecture notes set number 6.]

5. You will find in the class web-site [see matlab page], a sample set of seasonal farm employment data $(t_i, y_i)$ over about an 18 month period, where $t_i$ represents months, and $y_i$ is the employed population in millions. It is thought that this population, $y(t)$, evolves with time according to a function of the form:

$$y(t) = a_1 + a_2 t + a_3 \cos t.$$ 

Find $a_1, a_2, a_3$ by least-squares data fitting. Then plot the function you find. On the same plot show also the observed population [using a square for each point].