1. Implement a matlab function which computes the Householder QR factorization of a full-rank matrix \(A\). [You need to implement the “alternative” version which yields positive entries on the diagonal of the \(R\) matrix.] The main function should be as follows:

\[
[V, \beta] = \text{housQR}(A).
\]

Here \(V\) should contain the vectors \(v_1, \ldots, v_n\) related to the successive Householder reflectors \(P_j = I - \beta_j v_j v_j^T\) used to transform \(A\) into upper triangular form and \(\beta\) is the vector of the coefficients \(\beta_j\). The same array \(V\) also contains \(R\) matrix in its upper part. The basic scripts \text{house.m} and \text{house1.m} are provided in the class web-site (under matlab). The script \text{housQR} (a version of which was shown in class) does not generate the \(Q\) matrix. Along with it you need to have another script which generates the \(Q\) matrix from \(V\) and the array \(\beta\). Show the two matlab scripts. Apply these scripts to compute the QR factorization of the same matrix as the one you used for Question 3 of HW4. You can now compare the result of the Householder QR obtained in this way with the three you already compared in HW4. Show the same error measures \(\|A - Q^* R\|_2\), and \(\|I - Q^T Q\|_2\). Comment on what you observe relative to the results you had in HW4.

2. Consider the problem \(\min \|b - Ax\|_2\) in the situation where \(A\) is \(m \times n\) and \(m < n\) (called the ‘underdetermined’ case). Assume that \(A\) is of full rank. (a) Using what you learned from the URV decomposition, show that the set of solutions is of dimension \(n - m\). Show that the least-squares solution \(x^*\) of smallest norm must belong to \(\text{Ran}(A^T)\). (b) Find a method for computing \(x^*\) which involves a form of normal equations. (c) Find a method for computing \(x^*\) which involves the QR factorization. (d) Find a method for computing \(x^*\) based on the SVD of \(A\).

3. Use the method in part (c) of the previous question (via the QR method) to find the solution \(x^*\) with smallest norm of the linear system \(Ax = b\) where:

\[
A = \begin{pmatrix}
2 & -1 & 2 & 1 \\
2 & 1 & 2 & -1 \\
-2 & 1 & -1 & 0
\end{pmatrix}, \quad b = \begin{pmatrix}
5 \\
-1 \\
-3
\end{pmatrix}.
\]

[You can use matlab for the calculations and show a diary of an execution of your matlab commands]

4. Consider the matrix shown on the right: (a) What are the nonzero singular values of \(A\)?
(b) If \(A = U \Sigma V^T\) is the SVD decomposition of \(A\), what is the matrix \(V\)? What is \(\Sigma\)?
(c) Find the first two columns of the matrix \(U\).
(d) Find the matrix of rank 1, that is the closest to \(A\) in the 2-norm sense, i.e., the matrix \(A_1\) which minimizes \(\|A - B\|_2\) over all \(4 \times 2\) matrices \(B\) that are of rank 1.

5. Prove that if \(A \in \mathbb{R}^{m \times n}\) then

\[
\sigma_1(A) = \max_{y \neq 0 \in \mathbb{R}^n} \frac{y^T Ax}{\|y\|_2 \|x\|_2}
\]
[Hint: start by showing that $\sigma_1 \geq$ the right-hand side by using the Cauchy-Schwartz inequality]

6. We are given a square matrix $A$ that has the SVD $A = U\Sigma V^T$. Let $U = [u_1, u_2, \cdots, u_n]$, $V = [v_1, \cdots, v_n]$ and $\Sigma = \text{diag}(\sigma_1, \cdots, \sigma_n)$. Prove that the $2n$ eigenvalues of the matrix:

$$H = \begin{pmatrix} 0 & A^T \\ A & 0 \end{pmatrix}$$

are $\pm\sigma_i$ with the corresponding unit eigenvectors $\frac{1}{\sqrt{2}} \begin{bmatrix} v_i \\ \pm u_i \end{bmatrix}$. Extend to a general rectangular matrix $A$. 