1. This question involves a small matlab experiment. You will find in the matlab page of the class, the image of a “clown face” which is badly damaged by noise. [essentially adding large values to some random locations of the pixels.] Load the image with the command `load('clwn1')`. This will yield a matrix $X_1$ of pixels which can be viewed (see picture next) with the commands: `imagesc(X1) ; colormap('gray')`.

To smooth the image you will use matlab’s `svds` to compute the $k$-rank SVD approximation of the image and show the corresponding picture. Try a few values of $k$ between 10 and 50 and show 4 different results going from what you think is poor to better rendering of the image. Be aware that you will not get a perfect image but that you need to reach a compromise between noise and sharpness.

2. Let $u, v$ be two vectors in $\mathbb{R}^n$ that are orthonormal to each other. (a) Find the eigenvalues and eigenvectors of the following matrices:

$$A = uv^T - vu^T; \quad B = uv^T + vu^T; \quad C = uu^T + vv^T + \beta(uv^T + vu^T).$$

(b) Assuming that $[u, v]$ is an orthonormal system, find the thin SVD of the matrix $A$.

3. We have the Schur form of a certain matrix $T \in \mathbb{R}^{n \times n}$, as shown below in (a) and we want to find a matrix $S$ of the form shown in (b) below such that $S^{-1}TS$ has a zero column above the $(n, n)$ entry as shown in (c):

$$T = \begin{pmatrix} A & c \\ 0 & \beta \end{pmatrix}; \quad S = \begin{pmatrix} I \\ 0 \end{pmatrix}; \quad S^{-1}TS = \begin{pmatrix} A & 0 \\ 0 & \beta \end{pmatrix}.$$

Write the condition(s) required on $z$ in order to satisfy (c). State under what condition(s) can a $z$ be found and show how to find it in this case. Assume now that all the eigenvalues of $A$ are distinct. Show how you can transform $A$ in Schur form to diagonal form. What is the cost involved when it is doable?

4. Apply Gershgorin’s theorem to find a domain where the eigenvalues of $A$ are located for the following matrices

$$A_1 = \begin{pmatrix} 1 & -1 & -1 \\ 1 & 2 & 3 \\ 2 & -4 & 1 \end{pmatrix} \quad A_2 = \begin{pmatrix} 1-i & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 1-i & 1+i \end{pmatrix} \quad A_3 = \begin{pmatrix} 3 & i & -i \\ -i & 2 & 0 \\ i & 0 & 1 \end{pmatrix}.$$

The region (complex or real) you find should be the smallest possible that can be determined by using (the row version of) the Gershgorin theorem.

5. This is about the QR SVD algorithm. For an $m \times n$ matrix $A$ (assume $m > n$) the algorithm is as follows:
\[ A = Q_0 R_0 \]  \hspace{1cm} \text{// QR factorization of } A

For \( k = 0, 1, 2, \ldots \), Do:

\[ R_k^T = Q_{k+1} R_{k+1} \]  \hspace{1cm} \text{// QR factorization of } R_k^T

EndDo

[In the QR factorizations it is assumed that diagonal entries or \( R \) are positive]. If \( A = U \Sigma V^T \) is the 'thin' SVD of \( A \), then it is observed that the product \( Q_1 Q_3 \cdots Q_{2k+1} \ldots \) converges to \( V \), \( Q_2 Q_4 \cdots Q_{2k} \cdots \) converges to \( U \), and \( R_{2k} \) converges to \( \Sigma \).

(a) Try the algorithm on a random \( 6 \times 3 \) matrix.

(b) Show that \( Q_1 (R_1 R_0) \) is the QR decomposition of \( R_0^T R_0 \) (\( Q_1 \) is the Q factor, \( R_1 R_0 \) the R factor)

(c) Show that \( (R_1 R_0) Q_1 = R_1 R_1^T \) (Hint: focus first on the middle matrix \( R_0 \) on the left hand side) and then that \( R_1 R_1^T = R_2^T R_2 \). Going from \( A_0 = R_0^T R_0 \) to \( A_2 = R_2^T R_2 \) represents a step of the QR algorithm. Explain.

(d) More generally consider the matrices \( A_k = R_k^T R_k \) for \( k = 0, 2, 4, \ldots \). Show that these are the iterates of the QR algorithm applied to the matrix \( A_0 = R_0^T R_0 \).

6. The matrix shown on the right arises in boundary value problems with periodic boundary conditions.

(a) What is the inertia of this matrix? [Hint: See what an LU factorization gives]

(b) What does Gershgorin’s theorem give for this matrix?

(c) This matrix is almost tridiagonal. Is this structure preserved by the QR algorithm?

(d) Show a sequence of Givens rotations to transform this matrix into tridiagonal form \((T = QAQ^T \text{ where } Q \text{ is unitary}, T \text{ tridiagonal})\). What is the order of the number of operations required? Illustrate the process on a 5x5 example (pattern only - no values).

7. Consider a nonsymmetric real matrix \( A \) which has the Schur decomposition \( A = Q R Q^H \) where \( Q \) is unitary, and \( R \) upper triangular.

(a) Show that \( q_1 \) the first column of \( Q \) is an eigenvector of \( A \). Are the other columns eigenvectors? Find a Schur decomposition of the matrix \( A_1 = A - \sigma q_1 q_1^H \). What are the eigenvalues of \( A_1 \)?

(b) Assume that the eigenvalues of \( A \) are such that

\[ |\lambda_1| > |\lambda_2| > |\lambda_3| \geq |\lambda_4| \geq \cdots |\lambda_n| \]

where for simplicity it is assumed that the eigenvalues are all real. Suppose you use the power method and have computed the eigenpair \( \lambda_1, u_1 \). Suggest a method for computing the next eigenvalue \( \lambda_2 \) by the power method applied to a certain matrix. You will find that this method will compute a vector \( u_2 \) which is not an eigenvector of \( A \). How can you obtain an eigenvector \( u_2 \) of \( A \) associated with \( \lambda_2 \)? [Hint: Look for \( u_2 \) in the form of a linear combination of \( u_1 \) and \( w_2 \)].

8. Given a matrix \( A \) that is symmetric positive definite, find the inertia of the matrix

\[ B = \begin{pmatrix} I & XX^T \\ X & XX^T - A \end{pmatrix} \]
You may assume that $A$ is $n \times n$ and $X$ is $n \times m$ (so $B$ is $(n + m) \times (n + m)$). [Hint: you can use block Gaussian Elimination and ideas from the LDLT factorization to reduce $B$ to block-diagonal form].

9. For the matrix $A$ shown on the right, determine the $c, s$ pair of the Jacobi rotation needed to annihilate the entry (1,3) (and (3,1)) of $A$. Then apply the transformation.

$$ A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 0 \\ 3 & 0 & 0 \end{pmatrix} $$

10. To reduce a matrix into Hessenberg form by a similarity transformation it is most common to use Householder transforms (see notes). Show how you can use Givens rotations. You need to show the sequences of rotations $G(i, j)$ for the each row and in which order they are applied.