Solution of System:
\[
\begin{pmatrix}
5 & 10 & 25 \\
1 & 1 & 1 \\
0 & 10 & 25
\end{pmatrix}
\begin{pmatrix}
x_n \\
x_d \\
x_q
\end{pmatrix}
= 
\begin{pmatrix}
145 \\
12 \\
125
\end{pmatrix}
\]

Solution: You will find: \(x_n = 4\), \(x_d = 5\), \(x_q = 3\).

\(\text{3. } (A^T)^T =??\)  
Solution: \((A^T)^T = A\)

\(\text{4. } (AB)^T =??\)  
Solution: \((AB)^T = B^T A^T\)

\(\text{5. } (A^H)^H =??\)  
Solution: \((A^H)^H = A\)

\(\text{6. } (A^H)^T =??\)  
Solution: \((A^H)^T = \bar{A}\)

\(\text{7. } (ABC)^T =??\)  
Solution: \((ABC)^T = C^T B^T A^T\)

\(\text{8. } \text{True/False: } (AB)C = A(BC)\)  
Solution: \(\rightarrow \text{True}\)
9. True/False: \( AB = BA \)  
Solution: \( \rightarrow \) false

10. True/False: \( AA^T = A^T A \)  
Solution: \( \rightarrow \) false in general

12. Complexity? [number of multiplications and additions for matrix multiply]

Solution: Let \( A \in \mathbb{R}^{m \times n} \) and \( B \in \mathbb{R}^{n \times p} \). Then the product \( AB \) requires \( 2mnp \) operations (there are \( mp \) entries in all and each of them requires \( 2n \) operations).

13. What happens to these 3 different approaches to matrix-matrix multiplication when \( B \) has one column (\( p = 1 \))? 

Solution: In the first: \( C_{:,j} \) the \( j \)=th column of \( C \) is a linear combination of the columns of \( A \). This is the usual matrix-vector product.

In the second: \( C_{i,:} \) is just a number which is the inner product of the \( i \)th row of \( A \) with the column \( B \).

The 3rd formula will give the exact same expression as the first.

14. Characterize the matrices \( AA^T \) and \( A^T A \) when \( A \) is of dimension \( n \times 1 \).
Solution: When \( A \in \mathbb{R}^{n \times 1} \) then \( AA^T \) is a rank-one \( n \times n \) matrix and \( A^T A \) is a scalar: the inner product of the column \( A \) with itself. 

Show that \( A \in \mathbb{R}^{m \times n} \) is of rank one iff [if and only if] there exist two nonzero vectors \( u \in \mathbb{R}^m \) and \( v \in \mathbb{R}^n \) such that

\[
A = uv^T.
\]

What are the eigenvalues and eigenvectors of \( A \)?

Solution: (a)

First we show that: When both \( u \) and \( v \) are nonzero vectors then the rank of a matrix of the matrix \( A = uv^T \) is one. The range of \( A \) is the set of all vectors of the form

\[
y = Ax = uv^Tx = (v^Tx)u
\]

since \( u \) is a nonzero vector, and not all vectors \( v^Tx \) are zero (because \( v \neq 0 \)) then this space is
of dimension 1.

Next we show that: If $A$ is of rank one then there exist nonzero vectors $u, v$ such that $A = uv^T$. If $A$ is of rank one, then $\text{Ran}(A) = \text{Span}\{u\}$ for some nonzero vector $u$. So for every vector $x$, the vector $Ax$ is a multiple of $u$. Let $e_1, e_2, \cdots, e_n$ the vectors of the canonical basis of $\mathbb{R}^n$ and let $\nu_1, \nu_2, \cdots, \nu_n$ the scalars such that $Ae_i = \nu_i u$. Define $v = [\nu_1, \nu_2, \cdots, \nu_n]^T$. Then $A = uv^T$ because the matrices $A$ and $uv^T$ have the same columns. (Note that the $j$-th column of $A$ is the vector $Ae_j$). In addition, $v \neq 0$ otherwise $A = 0$ which would be a contradiction because $\text{rank}(A) = 1$.

(b) Eigenvalues /vectors

Write $Ax = \lambda x$ then notice that this means $(v^T x) u = \lambda x$ so either $v^T x = 0$ and $\lambda = 0$ or $x = u$ and $\lambda = v^T u$. Two eigenvalues: 0 and $v^T x$...
Is it true that

\[ \text{rank}(A) = \text{rank}(\bar{A}) = \text{rank}(A^T) = \text{rank}(A^H) \] ?

**Solution:**

The answer is yes and it follows from the fact that the ranks of \( A \) and \( A^T \) are the same and the ranks of \( A \) and \( \bar{A} \) are also the same.

It is known that \( \text{rank}(A) = \text{rank}(A^T) \). We now compare the ranks of \( A \) and \( \bar{A} \) (everything is considered to be complex).

The important property that is used is that if a set of vectors is linearly independent then so is its conjugate. [convince yourself of this by looking at material from 2033]. If \( A \) has rank \( r \) and for example its first \( r \) columns are the basis of the range, the the same \( r \) columns of \( \bar{A} \) are also linearly independent. So \( \text{rank}(\bar{A}) \geq \text{rank}(A) \). Now you can use a similar argument to show that \( \text{rank}(A) \geq \text{rank}(\bar{A}) \). Therefore the ranks are the same. \( \square \)
Eigenvalues of two similar matrices $A$ and $B$ are the same. What about eigenvectors?

Solution: If $Au = \lambda u$ then $XBX^{-1}u = \lambda u \rightarrow B(X^{-1}u) = \lambda(Xu) \rightarrow \lambda$ is an eigenvalue of $B$ with eigenvector $Xu$ (note the $Xu$ cannot be equal to zero because $u \neq 0$ and $X$ is nonsingular).

Given a polynomial $p(t)$ how would you define $p(A)$?

Solution: If $p(t) = \alpha_0 + \alpha_1 t + \alpha_2 t^2 + \cdots + \alpha_k t^k$ then

$$p(A) = \alpha_0 I + \alpha_1 A + \alpha_2 A^2 + \cdots + \alpha_k A^k$$

where

$$A^j = \underbrace{A \times A \times \cdots \times A}_{j \text{ times}}$$

Given a function $f(t)$ (e.g., $e^t$) how would you define $f(A)$? [You may limit yourself to the case when $A$ is diagonalizable]
Solution: The easiest way would be through the Taylor series expansion:

\[ f(A) = f(0)I + \frac{f'(0)}{1!}A + \frac{f''(0)}{2!}A^2 \cdots + \frac{f^{(k)}(0)}{k!}A^k + \cdots \]

However, this will require a justification: Will this expression ‘converge’ as the number of terms goes to infinity? This is where norms are useful. We will revisit this in next set.

24 If \( A \) is nonsingular what are the eigenvalues/eigenvectors of \( A^{-1} \)?

Solution: Assume that \( Au = \lambda u \). Multiply both sides by the inverse of \( A \): \( u = \lambda A^{-1}u \) - then by the inverse of \( \lambda \): \( \lambda^{-1}u = A^{-1}u \). Therefore, \( 1/\lambda \) is an eigenvalue and \( u \) is an associated eigenvector.

25 What are the eigenvalues/eigenvectors of \( A^k \) for a given integer power \( k \)?

Solution: Assume that \( Au = \lambda u \). Multiply both sides by \( A \) and repeat \( k \) times. You will get \( A^k u = \lambda^k u \). Therefore, \( \lambda^k \) is an eigenvalue of \( A^k \) and \( u \) is an associated eigenvector.

26 What are the eigenvalues/eigenvectors of \( p(A) \) for a polynomial \( p \)?
Solution: Using the previous result you can show that $p(\lambda)$ is an eigenvalue of $p(A)$ and $u$ is an associated eigenvector.

**Problem 27** What are the eigenvalues/eigenvectors of $f(A)$ for a function $f$? [Diagonalizable case]

Solution: This will require using the diagonalized form of $A$: $A = XD X^{-1}$. With this $f(A) = X f(D) X^{-1}$. It becomes clear that the eigenvalues are the diagonal entries of $f(D)$, i.e., the values $f(\lambda_i)$ for $i = 1, \cdots, n$. As for the eigenvectors - recall that they are the columns of the $X$ matrix in the diagonalized form – And $X$ is the same for $A$ and $f(A)$. So the eigenvectors are the same.

**Problem 28** For two $n \times n$ matrices $A$ and $B$ are the eigenvalues of $AB$ and $BA$ the same?

Solution: We will show that if $\lambda$ is an eigenvalue of $AB$ then it is also an eigenvalue of $BA$.

Assume that $ABu = \lambda u$ and multiply both sides by $B$. Then $BABu = \lambda Bu$ – which we write in the form: $BAv = \lambda v$ where $v = Bu$. In the situation when $v \neq 0$, we clearly see that $\lambda$ is a nonzero eigenvalue of $BA$ with the associated eigenvector $v$. We now deal with the case when $v = 0$. In this case, since $ABu = \lambda u$, and $u \neq 0$ we must have $\lambda = 0$. However,
clearly $\lambda = 0$ is also an eigenvalue of $BA$ because $\det(BA) = \det(AB) = 0$.

We can similarly show that any eigenvalue of $BA$ are also eigenvalues of $AB$ by interchanging the roles of $A$ and $B$. This completes the proof.

30 Trace, spectral radius, and determinant of $A = \begin{pmatrix} 2 & 1 \\ 3 & 0 \end{pmatrix}$.

Solution: Trace is 2, determinant is $-3$. Eigenvalues are 3, $-1$ so $\rho(A) = 3$.

31 What is the inverse of a unitary (complex) or orthogonal (real) matrix?

Solution: If $Q$ is unitary then $Q^{-1} = Q^H$.

32 What can you say about the diagonal entries of a skew-symmetric (real) matrix?

Solution: They must be equal to zero.

33 What can you say about the diagonal entries of a Hermitian (complex) matrix?

Solution: We must have $a_{ii} = \bar{a}_{ii}$. Therefore $a_{ii}$ must be real.
34  What can you say about the diagonal entries of a skew-Hermitian (complex) matrix?

Solution: We must have \( a_{ii} = -\bar{a}_{ii} \). Therefore \( a_{ii} \) must be purely imaginary.

35  Which matrices of the following type are also normal: real symmetric, real skew-symmetric, Hermitian, skew-Hermitian, complex symmetric, complex skew-symmetric matrices.

Solution: Real symmetric, real skew-symmetric, Hermitian, skew-Hermitian matrices are normal. Complex symmetric, complex skew-symmetric matrices are not necessarily normal.

39  What does the matrix-vector product \( V \alpha \) represent?

Solution: If \( \alpha = [a_0, a_2, \cdots, a_n] \) and \( p(t) \) is the \( n \)-th degree polynomial:

\[
p(t) = a_0 + a_1 t + a_2 t^2 + \cdots + a_n t^n
\]

then \( V \alpha \) is a vector whose components are the values \( p(x_0), p(x_1), \cdots, p(x_n) \).

40  Interpret the solution of the linear system \( V \alpha = y \) where \( \alpha \) is the unknown. Sketch a ‘fast’ solution method based on this.
Solution: Given the previous exercise, the interpretation is that we are seeking a polynomial of degree \( n \) whose values at \( x_0, \cdots, x_n \) are the components of the vector \( y \), i.e., \( y_0, y_1, \cdots, y_n \). This is known as polynomial interpolation (see csci 5302). The polynomial can be determined by, e.g., the Newton table in \( O(n^2) \) operations. \( \square \)