Unitary matrices preserve the 2-norm.

Solution: The proof takes only one line if we use the result $(Ax, y) = (x, A^H y)$:

$$\|Qx\|^2 = (Qx, Qx) = (x, Q^H Qx) = (x, x) = \|x\|^2.$$

When do we have equality in Cauchy-Schwarz?

Solution: From the proof of Cauchy-Schwarz it can be seen that we have equality when $x = \lambda y$, i.e., when they are colinear.

Expand $(x + y, x + y)$ – What does Cauchy-Schwarz imply?

Solution: You will see that you can derive the triangle inequality from this expansion and the Cauchy-Schwarz inequality.
Solution: Start by invoking the triangle inequality to write:

$$\|x\| = \|(x - y) + y\| \leq \|x - y\| + \|y\| \rightarrow \|x\| - \|y\| \leq \|x - y\|$$

Next exchange the roles of $x$ and $y$:

$$\|y\| - \|x\| \leq \|y - x\| = \|x - y\|$$

The two inequalities $\|x\| - \|y\| \leq \|x - y\|$ and $\|y\| - \|x\| \leq \|x - y\|$ yield the result since they imply that

$$-\|x - y\| \leq \|x\| - \|y\| \leq \|x - y\|$$

\[\square\]

Consider the metric $d(x, y) = \max_i |x_i - y_i|$. Show that any norm in $\mathbb{R}^n$ is a continuous function with respect to this metric.
Solution: We need to show that we can make \( \|y\| \) arbitrarily close to \( \|x\| \) by making \( y \) ‘close’ enough to \( x \), where ‘close’ is measured in terms of the infinity norm distance \( d(x, y) = \|x - y\|_\infty \).

Define \( u = x - y \) and write \( u \) in the canonical basis as \( u = \sum_{i=1}^{n} \delta_i e_i \). Then:

\[
\|u\| = \| \sum_{i=1}^{n} \delta_i e_i \| \leq \sum_{i=1}^{n} |\delta_i| \| e_i \| \leq \max |\delta_i| \sum_{i=1}^{n} \| e_i \|
\]

Setting \( M = \sum_{i=1}^{n} \| e_i \| \) we get

\[
\|u\| \leq M \max |\delta_i| = M \| x - y \|_\infty
\]

Let \( \epsilon \) be given and take \( x, y \) such that \( \| x - y \|_\infty \leq \frac{\epsilon}{M} \). Then, by using the second triangle inequality we obtain:

\[
| \|x\| - \|y\| | \leq \|x - y\| \leq M \max \delta_i \leq M \frac{\epsilon}{M} = \epsilon.
\]

This means that we can make \( \|y\| \) arbitrarily close to \( \|x\| \) by making \( y \) close enough to \( x \) in the sense of the defined metric. Therefore \( \| \cdot \| \) is continuous. \( \square \)

\( \square \)\hspace{1cm} In \( \mathbb{R}^n \) (or \( \mathbb{C}^n \)) all norms are equivalent.
Solution: We will do it for $\phi_1 = \|\cdot\|$ some norm, and $\phi_2 = \|\cdot\|_\infty$ [and one can see that all other cases will follow from this one].

1. Need to show that for some $\alpha$ we have $\|x\| \leq \alpha \|x\|_\infty$. Express $x$ in the canonical basis of $\mathbb{R}^n$ as $x = \sum x_i e_i$ [look up canonical basis $e_i$ from your csci2033 class.] Then

$$\|x\| = \|\sum x_i e_i\| \leq \sum |x_i| \|e_i\| \leq \max |x_i| \sum \|e_i\| = \|x\|_\infty \alpha$$

where $\alpha = \sum \|e_i\|$.

2. We need to show that there is a $\beta$ such that $\|x\| \geq \beta \|x\|_\infty$. Assume $x \neq 0$ and consider $u = x/\|x\|_\infty$. Note that $u$ has infinity norm equal to one. Therefore it belongs to the closed and bounded set $S_\infty = \{v|\|v\|_\infty = 1\}$. Since norms are continuous (seen earlier), the minimum of the norm $\|u\|$ for all $u$’s in $S_\infty$ is reached, i.e., there is a $u_0 \in S_\infty$ such that

$$\min_{u \in S_\infty} \|u\| = \|u_0\|.$$

Let us call $\beta$ this minimum value, i.e., $\|u_0\| = \beta$. Note in passing that $\beta$ cannot be equal to zero otherwise $u_0 = 0$ which would contradict the fact that $u_0$ belongs to $S_\infty$ [all vectors in $S_\infty$}
have infinity norm equal to one.] The result follows because \( u = x / \| x \|_\infty \), and so, remembering that \( u = x / \| x \|_\infty \), we obtain

\[
\left\| \frac{x}{\| x \|_\infty} \right\| \geq \beta \rightarrow \| x \| \geq \beta \| x \|_\infty
\]

This completes the proof \( \square \)

\( \varepsilon \) 8. Show that for any \( x \):

\[
\frac{1}{\sqrt{n}} \| x \|_1 \leq \| x \|_2 \leq \| x \|_1
\]

Solution: For the right inequality, it is easy to see that \( \| x \|_2 \leq \| x \|_1 \) because \( \sum_i x_i^2 \leq \left( \sum_i |x_i| \right)^2 \)

For the left inequality, we rely on Cauchy-Schwarz. If we call \( 1 \) the vector of all ones, then:

\[
\| x \|_1 = \sum_i |x_i| \cdot 1 \leq \| x \|_2 \| One \|_2 = \sqrt{n} \| x \|_2
\]

\( \square \)

\( \varepsilon \) 14. Show that \( \rho(A) \leq \| A \| \) for any matrix norm.
Solution: Let \( \lambda \) be the largest (in modulus) eigenvalue of \( A \) with associated eigenvector \( u \). Then

\[
Au = \lambda u \rightarrow \frac{\|Au\|}{\|u\|} = |\lambda| = \rho(A)
\]

This implies that

\[
\rho(A) \leq \max_{x \neq 0} \frac{\|Ax\|}{\|x\|} = \|A\|
\]

Given a function \( f(t) \) (e.g., \( e^t \)) how would you define \( f(A) \)? [You may limit yourself to the case when \( A \) is diagonalizable]

Solution: The easiest way would be through the Taylor series expansion.

\[
f(A) = f(0)I + \frac{f'(0)}{1!}A + \frac{f''(0)}{2!}A^2 \cdots \frac{f^{(k)}(0)}{k!}A^k + \cdots
\]

However, this will require a justification: Will this expression ‘converge’ as the number of terms goes to infinity? This is where norms are useful.

In the simplest case where \( A \) is diagonalizable you can write \( A = XDX^{-1} \) and then consider
the $k$-term part of the Taylor series expression above:

\[ F_k = f(0)I + \frac{f'(0)}{1!}A + \frac{f''(0)}{2!}A^2 + \cdots + \frac{f^{(k)}(0)}{k!}A^k \]

\[ = X \left[ f(0)I + \frac{f'(0)}{1!}D + \frac{f''(0)}{2!}D^2 + \cdots + \frac{f^{(k)}(0)}{k!}D^k \right] X^{-1} \]

\[ \equiv XD_kX^{-1} \]

where $D_k$ is the matrix inside the brackets in line 2 of above equations. The $i$ - th diagonal entry of $D_k$ is of the form

\[ f_k(\lambda_i) = f(0) + \frac{f'(0)}{1!}\lambda_i + \frac{f''(0)}{2!}\lambda_i^2 + \cdots + \frac{f^{(k)}(0)}{k!}\lambda_i^k, \]

which is just the $k$-term part of the Taylor series expansion of $f(\lambda_i)$. Each of these will converge to $f(\lambda_i)$. Now it is easy to complete the argument. If we call $D_f$ the diagonal matrix whose $i$-th diagonal entry is $f(\lambda_i)$ and $f_A$ the matrix defined by

\[ f_A = X D_f X^{-1}, \]
then clearly

\[ \| F_k - F_A \|_2 = \| X(D_k - D_A)X^{-1} \|_2 \leq \| X \|_2 \| X^{-1} \|_2 \| D_k - D_A \|_2 \]
\[ \leq \| X \|_2 \| X^{-1} \|_2 \max_i | f_k(\lambda_i) - f(\lambda_i) | \]

which converges to zero as \( k \) goes to infinity. \[
\]

17. The eigenvalues of \( A^H A \) and \( AA^H \) are real nonnegative.

**Solution:** Let us show it for \( A^H A \) [the other case is similar] If \( \lambda, u \) is an eigenpair of \( A^H A \) then \( (A^H A)u = \lambda u \). Take inner products with \( u \) on both sides. Then:

\[ \lambda(u, u) = ((A^H A)u, u) = (Au, Au) = \| Au \|^2 \]

Therefore, \( \lambda = \| Au \|^2 / \| u \|^2 \) which is a real nonnegative number. \[
\]

[Note: 1) Observe how simple the proof is for such an important fact. It is based on the result \( (Ax, y) = (x, A^H y) \). 2) The singular values of \( A \) are the square roots of the eigenvalues of \( A^H A \) if \( m \geq n \) or those of the eigenvalues of \( AA^H \) if \( m < n \). So there are always \( \min(m, n) \) singular values. This is really just a preliminary definition as we need to refer to singular values]
often – but we will see singular values and the singular value decomposition in great detail later.]

\[ \text{Prove that when } A = uv^T \text{ then } \|A\|_2 = \|u\|_2\|v\|_2. \]

**Solution:** We start by dealing with the eigenvalues of an arbitrary matrix of the form \( A = uv^T \) where both \( u \) and \( v \) are in \( \mathbb{R}^n \). From \( Ax = \lambda x \) we get:

\[
uv^Tx = \lambda x \rightarrow (v^Tx)u = \lambda x
\]

Notice that we did this because \( v^Tx \) is a scalar. We have 2 cases.

**Case 1:** \( v^Tx = 0 \). In this case it is clear that the equation \( Ax = \lambda x \) is satisfied with \( \lambda = 0 \). So any vector that is orthogonal to \( v \) is an eigenvector of \( A \) associated with the eigenvalue \( \lambda = 0 \). (It can be shown that the eigenvalue 0 is of multiplicity \( n - 1 \)).

**Case 2:** \( v^Tx \neq 0 \). In this case it is clear that the equation \( Ax = \lambda x \) is satisfied with \( \lambda = v^Tu \) and \( x = u \). So \( u \) is an eigenvector of \( A \) associated with the eigenvalue \( v^Tx \).

In summary the matrix \( uv^T \) has only two eigenvalues: 0, and \( v^Tu \).
Going back to the original question, we consider now $A = uv^T$ and we are interested in the 2-norm of $A$. We have

$$\|A\|_2^2 = \rho(A^TA) = \rho(vu^Tuv^T) = \|u\|_2^2 \rho(vv^T) = \|u\|_2^2 \|v\|_2^2.$$}

The last relation comes from what was done above to determine the eigenvalues of $vv^T$. So in the end, $\|A\|_2 = \|u\|_2 \|v\|_2$. □

19 In this case what is $\|A\|_F$?

**Solution:** Only the last part of the above answer changes ( $\rho$ is replaced by $Tr$ ) and you will find that actually the Frobenius norm of $uv^T$ is again equal to $\|u\|_2 \|v\|_2$. □