Unitary matrices preserve the 2-norm.

**Solution:** The proof takes only one line if we use the result \((Ax, y) = (x, A^H y)\):

\[
\|Qx\|^2 = (Qx, Qx) = (x, Q^H Qx) = (x, x) = \|x\|^2.
\]

When do we have equality in Cauchy-Schwarz?

**Solution:** From the proof of Cauchy-Schwarz it can be seen that we have equality when \(x = \lambda y\), i.e., when they are colinear.

Expand \((x + y, x + y)\) – What does Cauchy-Schwarz imply?

**Solution:** You will see that you can derive the triangle inequality from this expansion and the Cauchy-Schwarz inequality.
• Proof of the Hölder inequality.

\[ |(x, y)| \leq \|x\|_p \|y\|_q \text{, with } \frac{1}{p} + \frac{1}{q} = 1 \]

Proof: For any \(z_i, v_i\) all nonnegative we have, setting \(\zeta = \sum z_i\),

\[
\left( \sum \frac{z_i}{\zeta} v_i \right)^p \leq \sum \left( \frac{z_i}{\zeta} v_i^p \right) \text{ (convexity)} \rightarrow \\
\left( \sum z_i v_i \right)^p \leq \left[ \sum \left( \frac{z_i}{\zeta} v_i^p \right) \right] \zeta = \left[ \sum z_i v_i^p \right] \zeta^{p-1} \rightarrow \\
\sum z_i v_i \leq \left[ \sum z_i v_i^p \right]^{1/p} \zeta^{(p-1)/p} \\
\sum z_i v_i \leq \left[ \sum z_i v_i^p \right]^{1/p} \left[ \sum z_i \right]^{1/q}
\]

Now take \(z_i = x_i^q\), and \(v_i = y_i \ast x_i^{1-q}\). Then \(z_i v_i = x_i y_i\) and:

\[
z_i v_i^p = x_i^q \ast (y_i \ast x_i^{1-q})^p = y_i^p \ast x_i^{q+p-pq} = y_i^p \ast x_i^0 = y_i^p \]

\(\Box\)

\[\text{\textsf{2.0.5}}\] Second triangle inequality.

Solution: Start by invoking the triangle inequality to write:

\[ \|x\| = \|(x - y) + y\| \leq \|x - y\| + \|y\| \rightarrow \|x\| - \|y\| \leq \|x - y\| \]

2-2
Next exchange the roles of $x$ and $y$:

$$\|y\| - \|x\| \leq \|y - x\| = \|x - y\|$$

The two inequalities $\|x\| - \|y\| \leq \|x - y\|$ and $\|y\| - \|x\| \leq \|x - y\|$ yield the result since they imply that

$$-\|x - y\| \leq \|x\| - \|y\| \leq \|x - y\|$$

\[\square\]

\[\text{6} \] Consider the metric $d(x, y) = \max_i |x_i - y_i|$. Show that any norm in $\mathbb{R}^n$ is a continuous function with respect to this metric.

**Solution:** We need to show that we can make $\|y\|$ arbitrarily close to $\|x\|$ by making $y$ ‘close’ enough to $x$, where ‘close’ is measured in terms of the infinity norm distance $d(x, y) = \|x - y\|_\infty$. Define $u = x - y$ and write $u$ in the canonical basis as $u = \sum_{i=1}^{n} \delta_i e_i$. Then:

$$\|u\| = \|\sum_{i=1}^{n} \delta_i e_i\| \leq \sum_{i=1}^{n} |\delta_i| \|e_i\| \leq \max |\delta_i| \sum_{i=1}^{n} \|e_i\|$$
Setting \( M = \sum_{i=1}^{n} ||e_i|| \) we get \[ \|u\| \leq M \max \delta_i = M \|x - y\|_\infty \]

Let \( \epsilon \) be given and take \( x, y \) such that \( \|x - y\|_\infty \leq \frac{\epsilon}{M} \). Then, by using the second triangle inequality we obtain:

\[
|\|x\| - \|y\| | \leq \|x - y\| \leq M \max \delta_i \leq M \epsilon \leq M = \epsilon.
\]

This means that we can make \( \|y\| \) arbitrarily close to \( \|x\| \) by making \( y \) close enough to \( x \) in the sense of the defined metric. Therefore \( \|\cdot\| \) is continuous. \( \square \)

\textbf{7} In \( \mathbb{R}^n \) (or \( \mathbb{C}^n \)) all norms are equivalent.

**Solution:** We will do it for \( \phi_1 = \|\cdot\| \) some norm, and \( \phi_2 = \|\cdot\|_\infty \) [and one can see that all other cases will follow from this one].

1. Need to show that for some \( \alpha \) we have \( \|x\| \leq \alpha \|x\|_\infty \). Express \( x \) in the canonical basis of
\(\mathbb{R}^n\) as \(x = \sum x_i e_i\) [look up canonical basis \(e_i\) from your csci2033 class.] Then
\[
\|x\| = \|\sum x_i e_i\| \leq \sum |x_i| \|e_i\| \leq \max |x_i| \sum \|e_i\| = \|x\|_\infty \alpha
\]
where \(\alpha = \sum \|e_i\|\).

2. We need to show that there is a \(\beta\) such that \(\|x\| \geq \beta \|x\|_\infty\). Assume \(x \neq 0\) and consider \(u = x/\|x\|_\infty\). Note that \(u\) has infinity norm equal to one. Therefore it belongs to the closed and bounded set \(S_\infty = \{v|\|v\|_\infty = 1\}\). Since norms are continuous (seen earlier), the minimum of the norm \(\|u\|\) for all \(u\)'s in \(S_\infty\) is reached, i.e., there is a \(u_0 \in S_\infty\) such that
\[
\min_{u \in S_\infty} \|u\| = \|u_0\|.
\]
Let us call \(\beta\) this minimum value, i.e., \(\|u_0\| = \beta\). Note in passing that \(\beta\) cannot be equal to zero otherwise \(u_0 = 0\) which would contradict the fact that \(u_0\) belongs to \(S_\infty\) [all vectors in \(S_\infty\) have infinity norm equal to one.] The result follows because \(u = x/\|x\|_\infty\), and so, remembering that \(u = x/\|x\|_\infty\), we obtain
\[
\left\|\frac{x}{\|x\|_\infty}\right\| \geq \beta \rightarrow \|x\| \geq \beta \|x\|_\infty
\]
This completes the proof

Show that for any $x$:

$$\frac{1}{\sqrt{n}}\|x\|_1 \leq \|x\|_2 \leq \|x\|_1$$

Solution: For the right inequality, it is easy to see that $\|x\|_2 \leq \|x\|_1$ because $\sum_i x_i^2 \leq (\sum_i |x_i|)^2$

For the left inequality, we rely on Cauchy-Schwarz. If we call $\mathbf{1}$ the vector of all ones, then:

$$\|x\|_1 = \sum_i |x_i| \cdot \mathbf{1} \leq \|x\|_2 \|\mathbf{1}\|_2 = \sqrt{n}\|x\|_2$$

Show that $\rho(A) \leq \|A\|$ for any matrix norm.

Solution: Let $\lambda$ be the largest (in modulus) eigenvalue of $A$ with associated eigenvector $u$. Then

$$Au = \lambda u \rightarrow \frac{\|Au\|}{\|u\|} = |\lambda| = \rho(A)$$
This implies that

\[ \rho(A) \leq \max_{x \neq 0} \frac{\|Ax\|}{\|x\|} = \|A\| \]

\[ \square \]

\[ \square15 \] Given a function \( f(t) \) (e.g., \( e^t \)) how would you define \( f(A) \)? [You may limit yourself to the case when \( A \) is diagonalizable]

**Solution:** The easiest way would be through the Taylor series expansion.

\[
 f(A) = f(0)I + \frac{f'(0)}{1!}A + \frac{f''(0)}{2!}A^2 \cdots \frac{f^{(k)}(0)}{k!}A^k + \cdots
\]

However, this will require a justification: Will this expression 'converge' as the number of terms goes to infinity? This is where norms are useful.

In the simplest case where \( A \) is diagonalizable you can write \( A = XDX^{-1} \) and then consider the \( k \)-term part of the Taylor series expression above:
\[ F_k = f(0)I + \frac{f'(0)}{1!}A + \frac{f''(0)}{2!}A^2 + \cdots + \frac{f^{(k)}(0)}{k!}A^k \]

\[ = X\left[f(0)I + \frac{f'(0)}{1!}D + \frac{f''(0)}{2!}D^2 + \cdots + \frac{f^{(k)}(0)}{k!}D^k\right]X^{-1} \]

\[ \equiv XD_kX^{-1} \]

where \( D_k \) is the matrix inside the brackets in line 2 of above equations. The \( i \)-th diagonal entry of \( D_k \) is of the form

\[ f_k(\lambda_i) = f(0) + \frac{f'(0)}{1!}\lambda_i + \frac{f''(0)}{2!}\lambda_i^2 + \cdots + \frac{f^{(k)}(0)}{k!}\lambda_i^k, \]

which is just the \( k \)-term part of the Taylor series expansion of \( f(\lambda_i) \). Each of these will converge to \( f(\lambda_i) \). Now it is easy to complete the argument. If we call \( D_f \) the diagonal matrix whose \( i \)th diagonal entry is \( f(\lambda_i) \) and \( f_A \) the matrix defined by

\[ f_A = XD_fX^{-1}, \]
then clearly

\[ \|F_k - F_A\|_2 = \|X(D_k - D_A)X^{-1}\|_2 \leq \|X\|_2\|X^{-1}\|_2\|D_k - D_A\|_2 \]
\[ \leq \|X\|_2\|X^{-1}\|_2 \max_i |f_k(\lambda_i) - f(\lambda_i)| \]

which converges to zero as \( k \) goes to infinity. □

**\[ \] 17** The eigenvalues of \( A^H A \) and \( AA^H \) are real nonnegative.

**Solution:** Let us show it for \( A^H A \) [the other case is similar] If \( \lambda, u \) is an eigenpair of \( A^H A \) then \((A^H A)u = \lambda u\). Take inner products with \( u \) on both sides. Then:

\[ \lambda(u, u) = ((A^H A)u, u) = (Au, Au) = \|Au\|^2 \]

Therefore, \( \lambda = \|Au\|^2/\|u\|^2 \) which is a real nonnegative number. □

[Note: 1) Observe how simple the proof is for such an important fact. It is based on the result \((Ax, y) = (x, A^H y)\). 2) The singular values of \( A \) are the square roots of the eigenvalues of \( A^H A \) if \( m \geq n \) or those of the eigenvalues of \( AA^H \) if \( m < n \). So there are always \( \min(m, n) \) singular values. This is really just a preliminary definition as we need to refer to singular values]
Prove that when $A = uv^T$ then $\|A\|_2 = \|u\|_2\|v\|_2$.

**Solution:** We start by dealing with the eigenvalues of an arbitrary matrix of the form $A = uv^T$ where both $u$ and $v$ are in $\mathbb{R}^n$. From $Ax = \lambda x$ we get:

$$uv^T x = \lambda x \rightarrow (v^T x)u = \lambda x$$

Notice that we did this because $v^T x$ is a scalar. We have 2 cases.

**Case 1:** $v^T x = 0$. In this case it is clear that the equation $Ax = \lambda x$ is satisfied with $\lambda = 0$. So any vector that is orthogonal to $v$ is an eigenvector of $A$ associated with the eigenvalue $\lambda = 0$. (It can be shown that the eigenvalue 0 is of multiplicity $n - 1$).

**Case 2:** $v^T x \neq 0$. In this case it is clear that the equation $Ax = \lambda x$ is satisfied with $\lambda = v^T u$ and $x = u$. So $u$ is an eigenvector of $A$ associated with the eigenvalue $v^T x$.

In summary the matrix $uv^T$ has only two eigenvalues: 0, and $v^T u$. 

...
Going back to the original question, we consider now $A = uv^T$ and we are interested in the 2-norm of $A$. We have

$$\|A\|_2^2 = \rho(A^TA) = \rho(vu^Tuv^T) = \|u\|_2^2 \rho(vv^T) = \|u\|_2^2 \|v\|_2^2.$$

The last relation comes from what was done above to determine the eigenvalues of $vv^T$. So in the end, $\|A\|_2 = \|u\|_2 \|v\|_2$. □

In this case what is $\|A\|_F$?

Solution: Only the last part of the above answer changes ($\rho$ is replaced by $Tr$) and you will find that actually the Frobenius norm of $uv^T$ is again equal to $\|u\|_2 \|v\|_2$. □

Proof of Cauchy-Schwarz inequality:

$$(x, y)^2 \leq (x, x) (y, y). \quad (1)$$
Proof: We begin by expanding \((x - \lambda y, x - \lambda y)\) using properties of inner products:

\[(x - \lambda y, x - \lambda y) = (x, x) - \bar{\lambda}(x, y) - \lambda(y, x) + |\lambda|^2(y, y).\]

If \(y = 0\) then the inequality is trivially satisfied. Assume that \(y \neq 0\) and take \(\lambda = (x, y)/(y, y)\). Then, from the above equality, \((x - \lambda y, x - \lambda y) \geq 0\) shows that

\[
0 \leq (x - \lambda y, x - \lambda y) = (x, x) - 2\frac{|(x, y)|^2}{(y, y)} + \frac{|(x, y)|^2}{(y, y)}
= (x, x) - \frac{|(x, y)|^2}{(y, y)},
\]

which yields the result. \(\square\)