Exact solution of system

\[
\begin{pmatrix}
2 & 4 & 4 \\
1 & 5 & 6 \\
1 & 3 & 1
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2 \\
x_3
\end{pmatrix} =
\begin{pmatrix}
6 \\
4 \\
8
\end{pmatrix}
\]

Solution: You will find \( x = [1, 3, -2]^T \). □

Justify the column version of Back-substitution algorithm.

Solution: The system \( Ax = b \) can be written in column form as follows:

\[ x_1 a_{:,1} + x_2 a_{:,2} + \cdots + x_n a_{:,n} = b \]

In first step we compute \( x_n = b_n / a_{n,n} \). Now move last term in left-hand side of above system
to the right:

\[ x_1 a_{:,1} + x_2 a_{:,2} + \cdots + x_{n-1} a_{:,n-1} = b - x_n a_{:,n} \equiv b^{(1)} \]

This is a new system of \( n \) equations that has \( (n - 1) \) unknowns and the right-hand-side \( b^{(1)} \).
The last equation of this system is of the form \( 0 = 0 \) and can therefore be ignored. Thus, we end up with a system of size \( (n - 1) \times (n - 1) \) that is still upper triangular and we can repeat the above argument recursively. □

3 Exact operation count for GE.

Solution:

\[
T = \sum_{k=1}^{n-1} \left( \sum_{i=k+1}^{n} (2(n - k) + 3) \right) \\
= \sum_{k=1}^{n-1} (2(n - k) + 3)(n - k)
\]
\[ T = 2 \sum_{k=1}^{n-1} (n-k)^2 + 3 \sum_{k=1}^{n-1} (n-k) \]

\[ = 2 \sum_{j=1}^{n-1} j^2 + 3 \sum_{j=1}^{n-1} j \]

In the last step we made a change of variables \( j = n - k \). Now we know that \( \sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6} \) and \( \sum_{k=1}^{n} k = \frac{n(n+1)}{2} \) and so

\[ T = 2 \frac{(n-1)(n)(2n-1)}{6} + 3 \times \frac{n(n-1)}{2} \]

\[ = \ldots \]

\[ = n(n-1) \left( \frac{2n}{3} + \frac{7}{6} \right) \quad (1) \]

Finally observe the remarkable fact that the final expression (1) is always an integer (it has to be) no matter what (integer) value \( n \) takes. □

\[ \text{Practical use: Show how to use the LU factorization to solve linear systems with the same matrix } A \text{ and different } b \text{'s.} \]
**Solution:** If we have the LU factorization $A = LU$ available then we can solve the linear system $Ax = b$ by writing it as

$$L (Ux) = b$$

So we solve for $y$: $Ly = b$ then once $y$ is computed we solve for $x$: $Ux = y$. This involves two triangular solves at the cost of $n^2$ each instead of the $O(n^3)$ cost of redoing everything with Gaussian elimination.□

5. LU factorization of the matrix $A = \begin{pmatrix} 2 & 4 & 4 \\ 1 & 5 & 6 \\ 1 & 3 & 1 \end{pmatrix}$?

**Solution:** You will find

$$L = \begin{pmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 1/2 & 1/2 & 1 \end{pmatrix} \quad U = \begin{pmatrix} 2 & 4 & 4 \\ 0 & 2 & 4 \\ 0 & 0 & -3 \end{pmatrix}$$

6. Determinant of $A$?
**Solution:** It is the determinant of $U$ which is $-12$.

**True or false:** “Computing the LU factorization of matrix $A$ involves more arithmetic operations than solving a linear system $Ax = b$ by Gaussian elimination”.

**Solution:** The number of arithmetic operations is identical. (The LU factorization involves additional memory moves to store the factors - but these are no floating point operations)

**Operation count for Gauss-Jordan. Order of the cost? How does it compare with Gaussian Elimination?**

**Solution:** From the notes:

$$T = \sum_{k=1}^{n-1} \sum_{i=1}^{n-1} [2(n-k) + 3] = \sum_{k=1}^{n-1} (n-1)[2(n-k) + 3]$$

$$= (n-1) \sum_{j=1}^{n-1} [2j + 3]$$

$$= (n-1) [n(n-1) + 3(n-1)]$$

$$= (n-1)^2(n+3) = (n-1)^3 + 4(n-1)^2$$
The bottom line is that the cost is \( \approx n^3 \) which is 50% more expensive than GE. This additional cost is not worth it in spite of the simplicity of the algorithm. For this Gauss-Jordan is seldom used in practice. 

What is the matrix \( PA \) when

\[
P = \begin{pmatrix}
0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0
\end{pmatrix},
\quad
A = \begin{pmatrix}
1 & 2 & 3 & 4 \\
5 & 6 & 7 & 8 \\
9 & 0 & -1 & 2 \\
-3 & 4 & -5 & 6
\end{pmatrix}
\]

Solution: Instead of multiplying you just permute the row: row 1 in new matrix is row 3 of old
matrix, row 2 is row 1 of old matrix, etc.

\[
PA = \begin{pmatrix}
9 & 0 & -1 & 2 \\
1 & 2 & 3 & 4 \\
-3 & 4 & -5 & 6 \\
5 & 6 & 7 & 8
\end{pmatrix}
\]

\[10\] In the previous example where

>> A = [ 1 2 3 4; 5 6 7 8; 9 0 -1 2 ; -3 4 -5 6]

Matlab gives \( \text{det}(A) = -896 \). What is \( \text{det}(PA) \)?

**Solution:** It changes sign so \( \text{det}(PA) = 896 \). This is because the permutation \( \pi = [3, 1, 4, 2] \) is made of 3 interchanges.