1. Show that each $A_k$ [$A(1:k, 1:k)$ in matlab notation] is SPD.

Solution: Let $x$ be any vector in $\mathbb{R}^k$ and consider the vector $y$ of $\mathbb{R}^n$ obtained by stacking $x$ followed by $n - k$ zeros. Then it can be easily seen that: $(A_kx, x) = (Ay, y)$ and since $A$ is SPD then $(Ay, y) > 0$ and therefore $(A_kx, x) > 0$ for any $x$ in $\mathbb{R}^k$. Hence $A_k$ is SPD.

2. Consequence $\det(A_k) > 0$

Solution: This is because the determinant is the product of the eigenvalues which are real positive (see notes).

3. If $A$ is SPD then for any $n \times k$ matrix $X$ of rank $k$, the matrix $X^TAX$ is SPD.

Solution: For any $v \in \mathbb{R}^k$ we have $(X^TAXv, v) = (AXv, Xv)$. In addition, since $X$ is of full rank, then $Xv$ cannot be zero if $v$ is nonzero. Therefore we have $(AXv, Xv) > 0$. 
**Problem 4:** Show that if $A^T = A$ and $(Ax, x) = 0 \ \forall x$ then $A = 0$.

**Solution:** The condition implies that for all $x, y$ we have $(A(x + y), x + y) = 0$. Now expand this as: $(Ax, x) + (Ay, y) + 2(Ax, y) = 0$ for all $x, y$ which shows that $(Ax, y) = 0 \ \forall x, y$. This implies that $A = 0$ (e.g. take $x = e_j, y = e_i$)...

**Problem 5:** Show: A nonzero matrix $A$ is indefinite iff $\exists x, y : (Ax, x)(Ay, y) < 0$.

**Solution:**

Trivial. The matrix cant be PSD or NSD under the conditon

Need to prove: If $A$ is indefinite then there exist such that $x, y : (Ax, x)(Ay, y) < 0$.

Assume contrary is true, i.e., $\forall x, y(Ax, x)(Ay, y) \geq 0$. There is at least one $x_0$ such that $(Ax_0, x_0)$ is nonzero, otherwise $A = 0$ from previous question. Assume $(Ax_0, x_0) > 0$. Then $\forall y(Ax_0, x_0)(Ay, y) \geq 0$. which implies $\forall y : (Ay, y) \geq 0$, i.e., $A$ is positive semi-definite. This contradicts the assumption that $A$ is neither positive nor negative semi-defininte

**Problem 6:** The (standard) LU factorization of an SPD matrix $A$ exists.
Solution: This is an immediate consequence of the main theorem on existence (Lec. notes. set #5) and Exercise 1 in this set which showed that \( \det(A_k) = 0 \) for \( k = 1, \cdots, n \).

Example:

\[
A = \begin{pmatrix} 1 & -1 & 2 \\ -1 & 5 & 0 \\ 2 & 0 & 9 \end{pmatrix}
\]

Is \( A \) symmetric positive definite?

Solution: Answer is yes because \( \det(A_k) > 0 \) for \( k = 1, 2, 3 \).

What is the LDL\(^T\) factorization of \( A \) ?

Solution: The LDL\(^T\) factorization is:
\[
L = \begin{pmatrix}
1 & 0 & 0 \\
-1 & 1 & 0 \\
2 & 1/2 & 1 \\
\end{pmatrix}
\quad U = \begin{pmatrix}
1 & -2 & 1 \\
0 & 4 & 2 \\
0 & 0 & 4 \\
\end{pmatrix}
\]

Therefore \( A = LDLT \) where \( L \) is as given above and

\[
D = \begin{pmatrix}
1 & 0 & 0 \\
0 & 4 & 0 \\
0 & 0 & 4 \\
\end{pmatrix}
\]

(Note: The exact value of \( D \) is not provided.)

What is the Cholesky factorization of \( A \)?

**Solution:** From the above LDLT factorization we have \( A = GG^T \) with

\[
G = \begin{pmatrix}
1 & 0 & 0 \\
-1 & 2 & 0 \\
2 & 1 & 2 \\
\end{pmatrix}
\]