Show that \((I - \beta vv^T)x = \alpha e_1\) when \(v = x - \alpha e_1\) and \(\alpha = \pm \|x\|_2\).

**Solution:** Equivalent to showing that

\[x - (\beta x^Tv)v = \alpha e_1\] i.e., \(x - \alpha e_1 = (\beta x^Tv)v\)

but recall that \(v = x - \alpha e_1\) so we need to show that

\[\beta x^Tv = 1\] i.e., that \[\frac{2}{\|x - \alpha e_1\|_2^2} (x^Tv) = 1\]

\[\text{Denominator} = \|x\|_2^2 + \alpha^2 - 2\alpha e_1^Tx = 2(\|x\|_2^2 - \alpha e_1^Tx)\]

\[\text{Numerator} = 2x^Tv = 2x^T(x - \alpha e_1) = 2(\|x\|_2^2 - \alpha x^Te_1)\]

\[\text{Numerator/ Denominator} = 1. \square\]

Cost of Householder QR?
Solution: Look at the algorithm: each step works in rectangle $X(k : m, k : n)$. Step $k$ : twice

$$2(m - k + 1)(n - k + 1)$$

\[
T(n) = \sum_{k=1}^{n} 4(m - k + 1)(n - k + 1)
\]

\[
= 4 \sum_{k=1}^{n} [(m - n) + (n - k + 1)](n - k + 1)
\]

\[
= 4 \left[ (m - n) \times \frac{n(n + 1)}{2} + \frac{n(n + 1)(2n + 1)}{6} \right]
\]

\[
\approx (m - n) \times 2n^2 + \frac{4n^3}{3}
\]

\[
= 2mn^2 - \frac{2}{3}n^3
\]

Suppose you know the norms of each column of $X$ at the start. What happens to each of the norms of $X(2 : m, j)$ for $j = 2, \cdots, n$? Generalize this to step $k$ and obtain a procedure to inexpensively compute the desired norms at each step.
Solution: *The trick that is used is that* the 2-norm of each column does not change throughout the algorithm. *This is simple to see because each column is multiplied by a Householder transformation* $P_k$ *at each step. These Householder transformations are unitary and preserve the length. The square of the 2-norm of* $X(k : n, j)$ *(solid red lines in Figure)* *is the original square of the 2-norm of* $X(k : n, j)$ *minus the square of the 2-norm of* $X(1 : k - 1, j)$ *(dashed red lines in Figure)* *(solid red lines in Figure)* *In order to update* $\|X(k : n, j)\|^2$ *– all we have to do is subtract* $|X(k - 1, j)|^2$ *at each step* $k$. *This costs very little.*
Consider the mapping that sends any point $x$ in $\mathbb{R}^2$ into a point $y$ in $\mathbb{R}^2$ that is rotated from $x$ by an angle $\theta$. Find the matrix representing the mapping. [Hint: observe how the canonical basis is transformed.] Show an illustration. What is the mapping corresponding to an angle $-\theta$?

Solution: The vector $e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is transformed to $\begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix}$. The vector $e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ is transformed to $\begin{pmatrix} -\sin \theta \\ \cos \theta \end{pmatrix}$.

These are the first and second columns of the mapping! So the matrix representing the rotation is

$$R_\theta = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$$

An illustration is shown in the figure.

A Givens rotation performs a rotation of angle $-\theta$. 