Householder

[Note: \( \| \cdot \| = \| \cdot \|_2 \)]

\[ P = I - 2 w w^T \text{ with } \| w \| = 1 \]
or

\[ P = I - \beta v v^T \text{ with } \beta = 2 / v^T v \]

Question 1: given \( x \) find \( v \) s.t. \( Px = \alpha e_1 \)

\((I-\beta v v^T)x = \alpha e_1\)

Answer: \( v = x - \alpha e_1 \) with \( \alpha = \pm \| x \| \) [both signs work]

--- two different implementations

NEXT: generalization

Question 2: Given

\[ x = \begin{bmatrix} |x_1| \\ x_1 \in \mathbb{R}^k \\ x_2 \in \mathbb{R}^{n-k} \\ |x_2| \end{bmatrix} \]

find \( v \) such that \( P v = \begin{bmatrix} |x_1| \\ y \end{bmatrix} \) with \( y = \alpha e_1 \in \mathbb{R}^{n-k} \)

solution: select \( v \) as follows:

\[ v = \begin{bmatrix} |v_1| \\ |v_2| \end{bmatrix} \text{ set } v_1 = 0 \Rightarrow v = \begin{bmatrix} 0 \\ v_2 \end{bmatrix} \]

\[ Px =? \]

\[ x - \beta (v^T x) v = \begin{bmatrix} |x_1| \\ y \end{bmatrix} \]

scalar \( s \)

\[ y = x_2 - s \cdot v_2 \quad \text{ with } \quad s = \beta v_2^T x_2 \]

--- everything as if we work only on second part \( (x_2) \)

Obtain \( v_2 \) as a Householder vector to transform \( x_2 \) into \( \alpha e_1 \)

\[ X_1 = P_1 X \]
\[ X_2 = P_2 X_1 = P_1 P_1 X \]
\[ X_3 = P_3 X_2 = .... \]
\[ \vdots \]
\[ X_n = P_n X_{n-1} = P_n P_{n-1} .... P_1 X \quad \text{ upper triangular } \equiv \mathbb{R} \]
Apply inverse of $P_n P_{n-1} \ldots P_1$ on left:

$$[P_n P_{n-1} \ldots P_1]^{-1} = P_1^{-1} \times P_2^{-1} \ldots P_{n-1}^{-1} = P_1 P_2 \ldots P_n \equiv Q$$

$$[P_1^{-1} = P_1 ]$$

\[ X = Q \ R \]

$X$ is $m \times n$

Differences with Gram-Schmidt:
* here $Q$ is of size: $m \times m$
* $R$ is of size: $m \times n$ - $R$ is upper triangular.

How to solve LS problems?

Important: you never form $Q$ explicitly! [$m \times m$ matrix - expensive]

Want to min $|| Q \ R \ x - b ||$ == min $|| Q^\top (Q \ R \ x - b) || = \min || R \ x - Q^\top b ||$

\[
R = \begin{bmatrix}
R_1 \\
0
\end{bmatrix}, \quad Q^\top b = c = \begin{bmatrix}
c_1 \\
c_2
\end{bmatrix}
\]

\[
\| R_1 \ x - c_1 \|_2^2 = \| R_1 \ x - c_1 \|_2^2 + \| c_2 \|_2^2
\]

solve $R_1 \ x = c_1$ => Done