Domain decomposition-type methods

- Introduction – motivation
- Domain partitioning and distributed sparse matrices
- Basic algorithms: distributed Matvec
- Distributed preconditoners: additive Schwarz, multiplicative Schwarz.
- Schur complement techniques
- Graph partitionionsg
Introduction

- Thrust of parallel computing techniques in most applications areas.
- Programming model: Message-passing seems (MPI) dominates
- Open MP and threads for small number of processors
- Important new reality: parallel programming has penetrated the ‘applications’ areas [Sciences and Engineering + industry]
- Problem 1: algorithms lagging behind somewhat
- Problem 2: Message passing is painful for large applications. ‘Time to solution’ is high.
Domain Decomposition: A Model problem

**Problem:**

\[
\begin{align*}
\Delta u &= f \quad \text{in } \Omega \\
u &= u_\Gamma \quad \text{on } \Gamma = \partial \Omega.
\end{align*}
\]

**Domain:**

\[\Omega = \bigcup_{i=1}^{s} \Omega_i,\]

- Domain decomposition or substructuring methods attempt to solve a PDE problem (e.g.) on the entire domain from problem solutions on the subdomains \(\Omega_i\).
Discretization of domain
Coefficient Matrix
Types of mappings

(a) Vertex-based; (b) edge-based; and (c) element-based partitioning

- Can adapt PDE viewpoint to general sparse matrices
- Will use the graph representation and ’vertex-based’ viewpoint –
Generalization: Distributed Sparse Systems

- Simple illustration: Block assignment. Assign equation $i$ and unknown $i$ to a given 'process'
- Naive partitioning - won’t work well in practice
Best idea is to use the adjacency graph of $A$:

Vertices = \{1, 2, \cdots, n\};
Edges: $i \to j$ iff $a_{ij} \neq 0$

Graph partitioning problem:

- Want a partition of the vertices of the graph so that
  (1) partitions have $\sim$ the same sizes
  (2) interfaces are small in size
General Partitioning of a sparse linear system

$S_1 = \{1, 2, 6, 7, 11, 12\}$: This means equations and unknowns 1, 2, 3, 6, 7, 11, 12 are assigned to Domain 1.

$S_2 = \{3, 4, 5, 8, 9, 10, 13\}$

$S_3 = \{16, 17, 18, 21, 22, 23\}$

$S_4 = \{14, 15, 19, 20, 24, 25\}$
Alternative: Map elements / edges rather than vertices

Equations/unknowns 3, 8, 12 shared by 2 domains. From distributed sparse matrix viewpoint this is an overlap of one layer

Partitioners: Metis, Chaco, Scotch, ..

More recent: Zoltan, H-Metis, PaToH
➤ Standard dual objective: “minimize” communication + “balance” partition sizes

➤ Recent trend: use of hypergraphs [PaToh, Hmetis,...]
A few words about hypergraphs

- Hypergraphs are very general. Ideas borrowed from VLSI work.
- Main motivation: to better represent communication volumes when partitioning a graph. Standard models face many limitations.
- Hypergraphs can better express complex graph partitioning problems and provide better solutions.
- Example: completely nonsymmetric patterns ...
- Even rectangular matrices...
Example: \( V = \{1, \ldots, 9\} \) and \( E = \{a, \ldots, e\} \) with 
\( a = \{1, 2, 3, 4\}, \ b = \{3, 5, 6, 7\}, \ c = \{4, 7, 8, 9\}, \ d = \{6, 7, 8\}, \) and \( e = \{2, 9\} \)

\[
\begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\end{bmatrix}
\]
Distributed Sparse matrices (continued)

Once a good partitioning is found, questions are:

1. How to represent this partitioning?
2. What is a good data structure for representing distributed sparse matrices?
3. How to set up the various “local objects” (matrices, vectors, ..)
4. What can be done to prepare for communication that will be required during execution?
Local interface variables always ordered last.

Need: 1) to set up the various “local objects”. 2) Preprocessing to prepare for communications needed during iteration?
Local view of distributed matrix:

The local system:

\[
\begin{pmatrix}
B_i & F_i \\
E_i & C_i
\end{pmatrix}
\begin{pmatrix}
u_i \\
y_i
\end{pmatrix}
+ \begin{pmatrix} 0 \\
\sum_{j \in N_i} E_{ij} y_j
\end{pmatrix} = \begin{pmatrix} f_i \\
g_i
\end{pmatrix}
\]

\(u_i\): Internal variables; \(y_i\): Interface variables
The local matrix consists of 2 parts: a part (‘$A_{loc}$’) which acts on local data and another (‘$B_{ext}$’) which acts on remote data.

Once the partitioning is available these parts must be identified and built locally.

In finite elements, assembly is a local process.

How to perform a matrix vector product? [needed by iterative schemes?]
**Distributed Sparse Matrix-Vector Product Kernel**

**Algorithm:**

1. Communicate: exchange boundary data.

   Scatter $x_{\text{bound}}$ to neighbors - Gather $x_{\text{ext}}$ from neighbors

2. Local matrix – vector product

   $$ y = A_{\text{loc}}x_{\text{loc}} $$

3. External matrix – vector product

   $$ y = y + B_{\text{ext}}x_{\text{ext}} $$

**NOTE:** 1 and 2 are independent and can be overlapped.
Distributed Sparse Matrix-Vector Product

Main part of the code:

call MSG_bdx_send(nloc,x,y,nproc,proc,ix,ipr,ptrn,ierr)
c do local matrix-vector product for local points
call amux(nloc,x,y,alloc,jalloc,ialoc)
c receive the boundary information
call MSG_bdx_receive(nloc,x,y,nproc,proc,ix,ipr,ptrn,

c do local matrix-vector product for external points
nrow = nloc - nbnd + 1
call amux1(nrow,x,y(nbnd),alloc,jalloc,ialoc(nloc+1))
c return
The local exchange information

- List of adjacent processors (or subdomains)
- For each of these processors, lists of boundary nodes to be sent / received to / from adj. PE’s.
- The receiving processor must have a matrix ordered consistently with the order in which data is received.

Requirements

- The ‘set-up’ routines should handle overlapping
- Should use minimal storage (only arrays of size nloc allowed).
**Distributed Flexible GMRES (FGMRES)**

1. **Start:** Choose \( x_0 \) and \( m \). Let of the Krylov subspaces. Define \( \bar{H}_m \in \mathbb{R}^{(m+1) \times m} \) with \( \bar{H}_m \equiv 0 \). and initialize all its entries \( h_{i,j} \) to zero.

2. **Arnoldi process:**
   
   (a) Compute \( r_0 = b - Ax_0 \), \( \beta = \|r_0\|_2 \) and \( v_1 = r_0/\beta \).
   
   (b) For \( j = 1, \ldots, m \) do
   
   - Compute \( z_j := M_j^{-1}v_j \); Compute \( w := Az_j \);
   
   - For \( i = 1, \ldots, j \), do
     
     1. \( h_{i,j} := (w, v_i) \)
     
     2. \( w := w - h_{i,j}v_i \)
   
   - Compute \( h_{j+1,j} = \|w\|_2 \) and \( v_{j+1} = w/h_{j+1,j} \).

   (c) Define \( Z_m := [z_1, \ldots, z_m] \)

3. **Form the approximate solution:** Compute
   
   \( y_m = \arg\min_y \|\beta e_1 - \bar{H}_m y\|_2 \) and \( x_m = x_0 + [z_1, z_2, \ldots, z_m] y_m \)
and \( e_1 = [1, 0, \ldots, 0]^T \). with \( \bar{H}_m = \{h_{i,j}\}_{1 \leq i \leq j+1; 1 \leq j \leq m} \).

4. **Restart:** If satisfied stop, else set \( x_0 \leftarrow x_m \) and goto 1.
Main Operations in (F) GMRES:

1. Saxpy’s – local operation – no communication
2. Dot products – global operation
3. Matrix-vector products – local operation – local communication
4. Preconditioning operations – locality varies.
Distributed Dot Product

/*---------------------- call blas1 function */
tloc = DDOT(n, x, incx, y, incy);

/*---------------------- call global reduction */
MPI_Allreduce(&tloc,&ro,1,MPI_DOUBLE,MPI_SUM,comm);
A remark: the global viewpoint

\[
\begin{pmatrix}
B_1 \\
B_2 \\
\vdots \\
B_p
\end{pmatrix}
\begin{pmatrix}
F_1 \\
F_2 \\
\vdots \\
F_p
\end{pmatrix}
= \begin{pmatrix}
u_1 \\
u_2 \\
\vdots \\
u_p
\end{pmatrix}
\begin{pmatrix}
f_1 \\
f_2 \\
\vdots \\
f_p
\end{pmatrix}
\]

← Interior variables → ← Interface variables →
DISTRIBUTED PRECONDITIONERS
Example: Distributed ILU(0)

- Global view of matrix is (for 4 processors):

\[
A = \begin{pmatrix}
A_1 & A_2 & F_1 \\
A_2 & A_3 & F_2 \\
& A_3 & F_3 \\
E_1 & E_2 & E_3 & E_4 & F_4 \\
E_1 & E_2 & E_3 & E_4 & D
\end{pmatrix}
\]

- \(A_i\) = local matrix restricted to internal nodes only.

- 1-st approach: Idea: ILU on this matrix – parallelism available for diagonal blocks. Define an order in which to eliminate interface unknowns.
2-nd approach: Multi-color, \( k \)-step SOR or SSOR preconditioners.

3-rd approach: Solve equations for all interface points [Schur Complement approach] – to precondition, use ideas from DD.
Example: Distributed ILU(0) – cont.

- Easy to understand from a local view of distributed matrix
- Start by selecting an order [or a “schedule”, or a “priority rule”] in which to process globally
- Then locally:

  1. Eliminate internal rows
  2. Receive rows needed to process local interface rows
  3. Process local interface rows
  4. Send local interface rows to processors needing them
A distributed view of ILU(0) – schedule based on PE numbers

Note: any schedule can be used provided neighbors have different labels. Example: can use coloring.
A distributed view of ILU(0) – schedule based on PE coloring

See [S. Ma and YS. 1994]

Generalized ILU(k): D. Hysom and A. Pothen '00.

Used in pARMS for preconditioning Schur complement
Domain Decomposition–Type preconditioners

- Schwarz Preconditioners
- Schur-complement based Preconditioners
- Multi-level ILU-type Preconditioners

Observation: Often, in practical applications, Schwarz Preconditioners are used: SUB-OPTIMAL
Local view of distributed matrix:

\[
\begin{array}{ccc}
\text{External data} & \text{local Data} & \text{External data} \\
\hline
O & A_i & O \\
X_i & & X_i \\
\end{array}
\]

Block Jacobi Iteration (Additive Schwarz):
1. Obtain external data \( y_i \)
2. Compute (update) local residual
   \[
   r_i = (b - Ax)_i = b_i - A_i x_i - B_i y_i
   \]
3. Solve \( A_i \delta_i = r_i \)
4. Update solution \( x_i = x_i + \delta_i \)
Multiplicative Schwarz. Need a coloring of the subdomains so that:

- No two adjacent subdomains share same color

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**Multicolor Block SOR Iteration (Multiplicative Schwarz):**

1. **Do** \( col = 1, \ldots, numcols \)
2. If \((col \ eq. mycol)\) Then
3. Obtain external data \( y_i \)
4. Update local residual \( r_i = (b - Ax)_i \)
5. Solve \( A_i \delta_i = r_i \)
6. Update solution \( x_i = x_i + \delta_i \)
7. EndIf
8. EndDo
“Color” loop is sequential. Can be broken in several different ways.

(1) Have a few subdomains per processors
(2) Separate interior nodes from interface nodes (2-level blocking)

(3) Use a block-GMRES algorithm - with Block-size = number of colors. SOR step targets a different color on each column of the block ➤ no iddle time.
Local Solves

Each local system $A_i \delta_i = r_i$ can be solved in three ways:

1. By a (sparse) direct solver
2. Using a standard preconditioned Krylov solver
3. Doing a backward-forward solution associated with an accurate ILU (e.g. ILUT) preconditioner

We only use (2) with a small number of inner steps (up to 10) or (3).
SCHUR COMPLEMENT-BASED PRECONDITIONERS
Local system can be written as

$$A_i x_i + X_i y_{i, ext} = b_i.$$  \hfill (1)

$x_i$ = vector of local unknowns, $y_{i, ext}$ = external interface variables, and $b_i$ = local part of RHS.
Local equations

\[
\begin{pmatrix} B_i & F_i \\ E_i & C_i \end{pmatrix} \begin{pmatrix} u_i \\ y_i \end{pmatrix} + \left( \sum_{j \in N_i} E_{ij} y_j \right) = \begin{pmatrix} f_i \\ g_i \end{pmatrix}
\] (2)

eliminate \( u_i \) from the above system:

\[
S_i y_i + \sum_{j \in N_i} E_{ij} y_j = g_i - E_i B_i^{-1} f_i \equiv g'_i,
\]

where \( S_i \) is the “local” Schur complement

\[
S_i = C_i - E_i B_i^{-1} F_i.
\] (3)
**Structure of Schur complement system**

Global Schur complement system: 

\[ Sy = g' \text{ with:} \]

\[
S = \begin{pmatrix}
S_1 & E_{12} & \ldots & E_{1p} \\
E_{21} & S_2 & \ldots & E_{2p} \\
\vdots & \vdots & \ddots & \vdots \\
E_{p1} & E_{p-1,2} & \ldots & S_p \\
\end{pmatrix}
\begin{pmatrix}
y_1 \\
y_2 \\
\vdots \\
y_p \\
\end{pmatrix}
= \begin{pmatrix}
g'_1 \\
g'_2 \\
\vdots \\
g'_p \\
\end{pmatrix}.
\]

- \( E_{ij} \)'s are sparse = same as in the original matrix
- Can solve global Schur complement system iteratively. Back-substitute to recover rest of variables (internal).
- Can use the procedure as a preconditining to global system.
Simplest idea: Schur Complement Iterations

\[
\begin{pmatrix}
u_i \\
y_i
\end{pmatrix}
\]

- Internal variables
- Interface variables

- Do a global primary iteration (e.g., block-Jacobi)
- Then accelerate only the \( y \) variables (with a Krylov method)

Still need to precondition..
**Approximate Schur-LU**

Two-level method based on induced preconditioner. Global system can also be viewed as

\[
\begin{pmatrix}
B & F \\
E & C
\end{pmatrix}
\begin{pmatrix}
u \\
y
\end{pmatrix}
= \begin{pmatrix}f \\
g
\end{pmatrix}, \quad B =
\begin{pmatrix}
B_1 & B_2 & \cdots & B_p \\
E_1 & E_2 & \cdots & E_p \\
F_1 & F_2 & \cdots & F_p
\end{pmatrix}
\]

Block LU factorization of \(A\):

\[
\begin{pmatrix}
B & F \\
E & C
\end{pmatrix}
= \begin{pmatrix}
B & 0 \\
E & S
\end{pmatrix}
\begin{pmatrix}
I & B^{-1}F \\
0 & I
\end{pmatrix},
\]
Preconditioning:

\[
L = \begin{pmatrix} B & 0 \\ E & M_S \end{pmatrix} \quad \text{and} \quad U = \begin{pmatrix} I & B^{-1}F \\ 0 & I \end{pmatrix}
\]

with \( M_S = \) some approximation to \( S \).

- Preconditioning to global system can be induced from any preconditioning on Schur complement.

Rewrite local Schur system as

\[
y_i + S_i^{-1} \sum_{j \in N_i} E_{ij} y_j = S_i^{-1} \left[ g_i - E_i B_i^{-1} f_i \right].
\]

- equivalent to Block-Jacobi preconditioner for Schur complement.

- Solve with, e.g., a few \( s \) (e.g., 5) of GMRES
Question: How to solve with $S_i$?

Can use LU factorization of local matrix $A_i = \begin{pmatrix} B_i & F_i \\ E_i & C_i \end{pmatrix}$

and exploit the relation:

$$A_i = \begin{pmatrix} L_{B_i} & 0 \\ E_i U_{B_i}^{-1} & L_{S_i} \end{pmatrix} \begin{pmatrix} U_{B_i} & L_{B_i}^{-1} F_i \\ 0 & U_{S_i} \end{pmatrix} \Rightarrow L_{S_i} U_{S_i} = S_i$$

Need only the (l) LU factorization of the $A_i$ [rest is already available]

GRAPH PARTITIONING
Three approaches to graph partitioning:

1. Spectral methods (Recursive Spectral Bisection)
2. Geometric techniques [Houstis & Rice et al., Miller, Vavasis, Teng et al.] Coordinates are required.
3. Graph Theory techniques [use graph, but no coordinates]
   - Currently best known technique is Metis (multi-level algorithm)
   - Simplest idea: Recursive Graph Bisection; Nested dissection (George & Liu, 1980; Liu 1992)
   - Advantages: simplicity – no coordinates required
“Laplace-type” matrices associated with general undirected graphs – useful in many applications

- Given a graph $G = (V, E)$ define
  - A matrix $W$ of weights $w_{ij}$ for each edge
  - Assume $w_{ij} \geq 0$, $w_{ii} = 0$, and $w_{ij} = w_{ji}$ $\forall (i, j)$
  - The diagonal matrix $D = \text{diag}(d_i)$ with $d_i = \sum_{j \neq i} w_{ij}$

- Corresponding graph Laplacean of $G$ is:
  \[ L = D - W \]

- Gershgorin’s theorem $\rightarrow L$ is positive semidefinite
Graph Laplacians: Another viewpoint

Let $N$ be the incidence matrix: $N_{ij} = \pm 1$ if $i$-th edge is incident on the $j$-th vertex.

For example: A$\leftrightarrow$C,D, B$\leftrightarrow$D, C$\leftrightarrow$A, D$\leftrightarrow$A,B (undirected graph):

$$N = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & -1 & 0 & 1 \\ 0 & -1 & 0 & 2 \end{bmatrix},$$

yielding Laplacian = diagonal matrix of degrees — Adjacency matrix

$$N^T N = L = \begin{bmatrix} 2 & 0 & -1 & -1 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ -1 & -1 & 0 & 2 \end{bmatrix}.$$
Consider simplest case:

\[ w_{ij} = \begin{cases} 
1 & \text{if } (i, j) \in E \& i \neq j \\
0 & \text{else} 
\end{cases} \]

\[ D = \text{diag} \left[ d_i = \sum_{j \neq i} w_{ij} \right] \]

**Property:** (Graph partitioning) Consider situation when \( w_{ij} \in \{0, 1\} \). If \( x \) is a vector of signs (±1) then

\[ x^\top Lx = 4 \times (\text{‘number of edge cuts’}) \]

edge-cut = pair \((i, j)\) with \( x_i \neq x_j \)
Consequence: Can be used for partitioning graphs, or ‘clustering’ [take $p = \text{sign}(u_2)$, where $u_2 = 2$nd smallest eigenvector..]

Would like to minimize $(Lx, x)$ subject to $x \in \{-1, 1\}^n$ and $e^T x = 0$ [balanced sets]

Will solve a relaxed form of this problem

**Background:** Consider any symmetric (real) matrix $A$ with eigenvalues $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$ and eigenvectors $u_1, \cdots, u_n$

Recall that:
(Min reached for $x = u_1$)

\[
\min_{x \in \mathbb{R}^n} \frac{(Ax, x)}{(x, x)} = \lambda_1
\]

In addition:
(Min reached for $x = u_2$)

\[
\min_{x \perp u_1} \frac{(Ax, x)}{(x, x)} = \lambda_2
\]
For a graph Laplacean $u_1 = e =$ vector of all ones and

...vector $u_2$ is called the Fiedler vector. It solves a relaxed form of the problem -

$$\min_{x \in \{ -1, 1 \}^n; \; e^T x = 0} \frac{(Lx, x)}{(x, x)} \rightarrow \min_{x \in \mathbb{R}^n; \; e^T x = 0} \frac{(Lx, x)}{(x, x)}$$

Define $v = u_2$ then $lab = sign(v - med(v))$
**Normalized Graph Cuts**

Mark a partitioning of the vertices: \( n_- = 1, \ n_+ = 3 \)

\[
v = \begin{bmatrix} 1, 1, 1, -3 \end{bmatrix}^T / \sqrt{3} \cdot 1 = [n_-, n_-, n_-, -n_+]^T / \sqrt{n_- n_+}.
\]

Then

\[
\frac{v^T Lv}{v^Tv} = |\text{cut}| \cdot \left( \frac{1}{n_-} + \frac{1}{n_+} \right)
\]

and \( v^T e = 0 \), where \( e = [1, 1, 1, 1]^T = \text{eigenvector of} \ L. \)

Approximately minimize this with an eigenvector of \( L \):

- \(-1.0 \cdot 10^{-15} (\ 0.500000 \ 0.500000 \ 0.500000 \ 0.500000) \) ← ‘null’ vector
- \(0.585786 (-.27059 \ 0.653281 \ -.65328 \ 0.270598) \) ← ‘Fiedler’
- \(2.00000 (\ 0.500000 \ -.500000 \ -.500000 \ 0.500000) \) vector
- \(3.41421 (\ 0.653281 \ .270598 \ -.27059 \ -.65328) \)
The Level Set Expansion Algorithm

➤ Given: \( p \) nodes ‘uniformly’ spread in the graph (roughly same distance from one another).

Do a level-set traversal from each node simultaneously.

Best described for an example on a \( 15 \times 15 \) five – point Finite Difference grid.

➤ See [Goehring-Saad ’94, See Cai-Saad ’95]