The equality & inequality constrained optimization problem

minimize $f(x)$
subject to $h_i(x) = 0 \quad i = 1, \ldots, m$
$g_j(x) \leq 0 \quad j = 1, \ldots, n$
Constrained Optimization

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- Domain \( \mathcal{D} = \text{dom}(f) \cap \bigcap_{i=1}^m \text{dom}(h_i) \cap \bigcap_{j=1}^n \text{dom}(g_j) \)
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The Lagrangian

\[ L(x, \lambda, \nu) = f(x) + \lambda^T h(x) + \nu^T g(x) \]

\[ = f(x) + \sum_{i=1}^{m} \lambda_i h_i(x) + \sum_{j=1}^{n} \nu_j g_j(x) \]
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Domain \( \text{dom}(L) = \mathcal{D} \times \mathbb{R}^m \times \mathbb{R}^n \)
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\text{minimize } f(x) \\
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Domain \( D = \text{dom}(f) \cap \bigcap_{i=1}^{m} \text{dom}(h_i) \cap \bigcap_{j=1}^{n} \text{dom}(g_j) \)

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Domain \( \text{dom}(L) = D \times \mathbb{R}^m \times \mathbb{R}^n \)

\( \{\lambda_i\}_{i=1}^{m}, \{\nu_j\}_{j=1}^{n} \) are the Lagrange multipliers
The Lagrange dual function

\[ L^*(\lambda, \nu) = \inf_{x \in D} L(x, \lambda, \nu) = \inf_{x \in D} \left( f(x) + \sum_{i=1}^{m} \lambda_i h_i(x) + \sum_{j=1}^{n} \nu_j g_j(x) \right) \]
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Let \( p^* \) be the constrained optimum of \( f(x) \)

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Lagrange Dual

- The Lagrange dual function
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- The Lagrange dual \( L^* \) is
  - A concave function, even when original problem is not convex
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How close is the maximum of \( L^*(\lambda, \nu) \) to \( p^* \)?
An Example

minimize \( x^T x \)
subject to \( A x = b \)

- Lagrangian \( L(x, \lambda) = x^T x + \lambda^T (A x - b) \)
- Recall that \( L^*(\lambda) = \inf_x L(x, \lambda) \)
- Setting gradient to 0, \( x = -\frac{1}{2} A^T \lambda \)
- Hence, the dual

\[
L^*(\lambda) = L\left(-\frac{1}{2} A^T \lambda, \lambda\right) = -\frac{1}{4} \lambda^T A A^T \lambda - \lambda^T b
\]

- \( L^*(\lambda) \) is a lower bounding concave function
The Lagrange Dual Problem

maximize $L^*(\lambda, \nu)$
subject to $\nu \geq 0$

- Best lower bound to $p^*$, the optimal of the primal
The Lagrange Dual Problem

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- Best lower bound to $p^*$, the optimal of the primal
- Concave optimization problem with maximum $d^*$
- Constraints are $\nu \geq 0$ and $(\lambda, \nu) \in \text{dom}(L^*)$
- For example, in linear programming

minimize $c^T x$
subject to $Ax = b$
$x \geq 0$

maximize $-b^T \lambda$
subject to $A^T \lambda + c \geq 0$
Weak and Strong Duality

- Weak Duality: $d^* \leq p^*$

Strong Duality:

- $d^* = p^*$
  - Does not hold in general
  - If it holds, it is sufficient to solve the dual
  - How to check if it holds?

Constraint Qualification

- Normally true on convex problems
- True if the convex problem is strictly feasible, e.g.,
  $\exists x \in \text{relint}(D)$ s.t.
  $Ax = b, g_j(x) < 0, \text{for some } j$

Slater's Condition for strong duality
Weak and Strong Duality

- **Weak Duality**: $d^* \leq p^*$
  - Always holds

Non-trivial lower bounds for hard problems
Used in approximation algorithms

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- Slater’s Condition for strong duality
Example: Quadratic Programs

\[
\begin{align*}
\text{minimize} & \quad x^T x \\
\text{subject to} & \quad Ax \leq b
\end{align*}
\]

- Lagrange dual

\[
L^*(\nu) = \inf_x \left( x^T x + \nu^T (Ax - b) \right) = -\frac{1}{4} \nu^T AA^T \nu - b^T \nu
\]

- Dual problem

\[
\begin{align*}
\text{maximize} & \quad -\frac{1}{4} \nu^T AA^T \nu - b^T \nu \\
\text{subject to} & \quad \nu \geq 0
\end{align*}
\]

- From Slater’s condition, \( p^* = d^* \)
- It is sufficient to solve the dual
If strong duality holds, $\mathbf{x}^*$ for primal, $(\lambda^*, \nu^*)$ for dual

$$f(\mathbf{x}^*) = L^*(\lambda^*, \nu^*) = \inf_{\mathbf{x}} \left( f(\mathbf{x}) + \sum_{i=1}^{m} \lambda_i^* h_i(\mathbf{x}) + \sum_{j=1}^{n} \nu_j^* g_j(\mathbf{x}) \right)$$

$$\leq f(\mathbf{x}^*) + \sum_{i=1}^{m} \lambda_i^* h_i(\mathbf{x}^*) + \sum_{j=1}^{n} \nu_j^* g_j(\mathbf{x}^*)$$

$$\leq f(\mathbf{x}^*)$$
Complementary Slackness

- If strong duality holds, $x^*$ for primal, $(\lambda^*, \nu^*)$ for dual

$$f(x^*) = L^*(\lambda^*, \nu^*) = \inf_x \left( f(x) + \sum_{i=1}^{m} \lambda_i^* h_i(x) + \sum_{j=1}^{n} \nu_j^* g_j(x) \right)$$

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- The two inequalities hold with equality
Complementary Slackness

- If strong duality holds, $\mathbf{x}^*$ for primal, $(\lambda^*, \nu^*)$ for dual

\[
\begin{align*}
    f(\mathbf{x}^*) &= L^*(\lambda^*, \nu^*) = \inf_{\mathbf{x}} \left( f(\mathbf{x}) + \sum_{i=1}^{m} \lambda_i^* h_i(\mathbf{x}) + \sum_{j=1}^{n} \nu_j^* g_j(\mathbf{x}) \right) \\
    &\leq f(\mathbf{x}^*) + \sum_{i=1}^{m} \lambda_i^* h_i(\mathbf{x}^*) + \sum_{j=1}^{n} \nu_j^* g_j(\mathbf{x}^*) \\
    &\leq f(\mathbf{x}^*)
\end{align*}
\]

- The two inequalities hold with equality
  - $\mathbf{x}^*$ minimizes the Lagrangian $L(\mathbf{x}, \lambda^*, \nu^*)$
If strong duality holds, $x^*$ for primal, $(\lambda^*, \nu^*)$ for dual

$$f(x^*) = L^*(\lambda^*, \nu^*) = \inf_x \left( f(x) + \sum_{i=1}^{m} \lambda_i^* h_i(x) + \sum_{j=1}^{n} \nu_j^* g_j(x) \right)$$

$$\leq f(x^*) + \sum_{i=1}^{m} \lambda_i^* h_i(x^*) + \sum_{j=1}^{n} \nu_j^* g_j(x^*)$$

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The two inequalities hold with equality

- $x^*$ minimizes the Lagrangian $L(x, \lambda^*, \nu^*)$
- $\nu_j^* g_j(x^*) = 0$ for all $j = 1, \ldots, n$ so that

$$\nu_j^* > 0 \Rightarrow g_j(x^*) = 0, \quad \text{and} \quad g_j(x^*) < 0 \Rightarrow \nu_j^* = 0$$
Karush-Kuhn-Tucker (KKT) Conditions

Necessary conditions satisfied by any primal and dual optimal pairs \( \tilde{x} \) and \( (\tilde{\lambda}, \tilde{\nu}) \)

- **Primal Feasibility:**
  
  \[
  h_i(\tilde{x}) = 0, \ i = 1, \ldots, n, \quad g_j(\tilde{x}) \leq 0, \ j = 1, \ldots, m
  \]
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- **Gradient condition:**
  \[ \nabla f(\tilde{x}) + \sum_{i=1}^{n} \tilde{\lambda}_i \nabla h_i(\tilde{x}) + \sum_{j=1}^{m} \tilde{\nu}_j \nabla g_j(\tilde{x}) = 0 \]
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  \nabla f(\tilde{x}) + \sum_{i=1}^{n} \tilde{\lambda}_i \nabla h_i(\tilde{x}) + \sum_{j=1}^{m} \tilde{\nu}_j \nabla g_j(\tilde{x}) = 0
  \]

- The conditions are sufficient for a convex problem.

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Constrained Optimization, Duality