Assignment #1: Number Representation on a Computer, Loss of Precision, Systems of Linear Equations, Echelon Form

Due date: Monday, February 2, 2015 (4:00pm)

Name: ____________________________________________________________

Section Number
Assignment #1: Number Representation on a Computer, Loss of Precision, Systems of Linear Equations, Echelon Form  

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For full credit you must show all of your work.

1. Suppose you had 5 bits to use to represent a number on a computer.  
   a) How many different numbers could you represent (using a single encoding scheme)?  
   b) Give the decimal representation of each of the numbers you could represent using 5 binary bits and the following encoding schemes. If the numbers are evenly spaced, it is sufficient to give the range (minimum, maximum) and the interval between the representable numbers within that range.  
      i) unsigned integer  
      ii) signed integer (please specify any assumptions you make about the specific encoding scheme)  
      iii) fraction (.xxxxx in binary)  
      iv) fixed point binary with 2 digits after the binary point (xxx.xx in binary)  
      v) floating point with 2 digits used to represent a biased exponent (using –1 as the biasing amount) and a 3 digit normalized mantissa (1.xxx in binary). For example, 00000 \rightarrow 2^{0-1} \times (1.00)_2 = 0.5

2. Which of the following numbers cannot be exactly represented on a computer as a 32-bit floating point number encoded in binary using the IEEE Standard for Floating Point Arithmetic (IEEE 754)?  
   a) 1/2  
   b) 1/3  
   c) 1/5  
   d) 1/8  
   e) 1/10

3. The program example0.m, shown below and written in Matlab, attempts to numerically compute \( y = \frac{d}{dx}(\sin x) \) using the classical definition \( f'(x) = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h} \) for increasingly smaller values of \( h \).

   ```matlab
   n = 30;
   x = 0.5;
   h = zeros(n,1);
   y = zeros(n,1);
   err = zeros(n,1);
   h(1) = 1;
   for i = 1:n-1
       y(i) = (sin(x + h(i)) - sin(x))/h(i)
       err(i) = abs(cos(x) - y(i))
       h(i+1) = h(i)/4;
   end
   ```

   Use Matlab to run this program. What trend do you observe in the values of \( y \)? Comparing successive estimates of \( y \) to the exact solution \( \frac{d}{dx}(\sin x) = \cos x \), how does the error change as \( h \) gets smaller? What happens in the limit as \( h \) goes to zero? Explain these results.

   Please attach a printout of the last page (only) of the values you get for \( y \) and \( \text{err} \).
4. Given an example of a system of linear equations in three variables that has:
   a) a unique solution
   b) no solution
   c) an infinite number of solutions
In each case, express your answer in the form: \( a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1 \), etc. where each of the \( a_{ij} \) and \( b_j \) are non-zero.

5. Do the two lines described by the equations below intersect? If so, at what point?
   \[
   \begin{align*}
   3x + 6y &= -3 \\
   5x + 7y &= 10
   \end{align*}
   \]

6. Which of the following systems, represented here by \textbf{augmented matrices}, are \textit{consistent}? Explain your answer.
   \[
   \begin{align*}
   \text{a)} & \begin{bmatrix} 1 & 7 & 3 & -4 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 1 & -2 \end{bmatrix} & \text{b)} & \begin{bmatrix} 1 & 7 & 3 & -4 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & -2 \end{bmatrix} & \text{c)} & \begin{bmatrix} 1 & 1 & 1 & 3 \\ 0 & 2 & 1 & 2 \\ 0 & 0 & 1 & 0 \end{bmatrix}
   \end{align*}
   \]

7. Is \((1, 2, 3)\) a solution of this system:
   \[
   \begin{bmatrix} 3 & 2 & 1 & 10 \\ 2 & 3 & 0 & 8 \\ 5 & 5 & 1 & 18 \end{bmatrix}
   \]

8. For what values of \( h \) is this system consistent:
   \[
   \begin{bmatrix} 1 & h & -5 \\ 2 & -8 & 6 \end{bmatrix}
   \]

9. Suppose \( a, b, c, \) and \( d \) are constants such that \( a \) is not zero and the system below is consistent for all possible values of \( f \) and \( g \). What relationship must hold among the numbers \( a, b, c, \) and \( d \)?
   \[
   \begin{align*}
   ax_1 + bx_2 &= f \\
   cx_1 + dx_2 &= g
   \end{align*}
   \]

10. Show, step-by-step, how to reduce the following augmented matrix into \textit{echelon} form. \textit{For full credit, you must provide enough detail to allow the grader to follow your work.}
   \[
   \begin{bmatrix}
   0 & 2 & 3 & 3 \\
   2 & 3 & 1 & 5 \\
   1 & 0 & 2 & 4
   \end{bmatrix}
   \]
   Is this system \textit{consistent}? You do not need to solve it (yet) to answer that question.
11. Show, step-by-step, how to reduce the following augmented matrix into reduced echelon form. For full credit, you must provide enough detail to allow the grader to follow your work.

\[
\begin{bmatrix}
2 & 3 & 1 & 5 \\
0 & 2 & 3 & 3 \\
0 & 0 & 1 & 1
\end{bmatrix}
\]

Does this system have a unique solution?

12. Use Matlab to convert the following augmented matrices into reduced echelon form. What is the solution to each of these systems?

\[
\begin{bmatrix}
2 & 2 & 9 & 7 \\
1 & 0 & -3 & 8 \\
0 & 1 & 5 & -2
\end{bmatrix}
\begin{bmatrix}
0 & 2 & 3 & 3 \\
2 & 3 & 1 & 5 \\
1 & 0 & 2 & 4
\end{bmatrix}
\begin{bmatrix}
3 & -3 & 1 & 0 \\
-2 & 2 & 5 & 0 \\
1 & -1 & -2 & 0
\end{bmatrix}
\begin{bmatrix}
1 & -4 & 7 & 2 \\
0 & 3 & -5 & 1 \\
-2 & 5 & -9 & 0
\end{bmatrix}
\]

13. Find the solution to the following system (indicated by the augmented matrix below) and express it in parametric form. What is the geometric description of this solution in \( \mathbb{R}^3 \)?

\[
\begin{bmatrix}
3 & -3 & 1 & 0 \\
-2 & 2 & 5 & 0 \\
1 & -1 & -2 & 0
\end{bmatrix}
\]

14. A system of linear equations that has fewer equations than unknowns is sometimes called an undetermined system. Can such a system have a unique solution? Explain.

15. A system of linear equations that has more equations than unknowns is sometimes called an overdetermined system. Can such a system be consistent? Explain.